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METHODS OF
CORRELATION ANALYSIS

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METHODS OF CORRELATION ANALYSIS

BY

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PREFACE

This book is not intended to cover the entire field of statistics, but rather, as its name indicates, that part of the field which is concerned with studying the relations between variables. The first two chapters are devoted to a brief review of the central elements in the measurement of variability in a statistical series, and to the essential concepts in judging the reliability of conclusions. These chapters are not to be regarded as a full statement, but instead as brief summaries to clarify the basic ideas which are involved in the subsequent development.

No attempt is made in the body of the text to present the mathematical theory on which the art of statistical analysis is based. Instead, the aim throughout has been to show how the various methods may be employed in practical research work, what their limitations are, and what the results really mean. Only the simplest of algebraic statements have been employed, and the practical procedure for each operation has been worked out step by step. It is believed that the material will be readily comprehensible to anyone who has had courses in elementary algebra.

Although the examples which are used in presenting the several methods are drawn very largely from the author's own field of agricultural economics, the methods themselves are explained in sufficiently general terms so that they can be applied in any field. In addition, two chapters are devoted to a discussion of the types of problems in a great many different fields of work to which correlation analysis has been successfully applied, and to research methods and the place of correlation analysis in research. It is hoped that this presentation will assist research workers in many fields to appreciate both the possibilities and the limitations of correlation analysis, and so gain from their data knowledge of all the relations which so frequently lie hidden beneath the surface.

Where the methods presented are the well-established ones developed by the fathers of the modern science, mainly the English statisticians, no attempt is made to prove or derive the various formulas. On a few crucial points, however, or where derivations not generally accessible are involved, the derivations of the formulas are shown in notes in the technical appendix, in the simplest manner possible.

The methods presented in this book, insofar as they constitute an advance over those previously available, represent largely the joint product of a group of young researchers in the Bureau of Agricultural Economics of the United States Department of Agriculture during the past decade. The new methods include (a) the application of the Doolittle method to the solution of multiple correlation problems, greatly reducing the labor of obtaining multiple correlation results, and making feasible the use of multiple correlation in actual research work; (b) the development of approximate methods for determining curvilinear multiple correlations, and, more recently, very rapid graphic methods for their determination; (c) the recognition of "joint" correlation, and the gradual development of methods of treating it; and (d) by extensive use in actual investigations, concrete demonstration of the possibilities of these methods in research work. These recent developments in correlation analysis are as yet largely unavailable except in the original articles in technical journals. One object of this book is to present them in organized form, and with such interpretation that their significance and application may be fully understood.

During the last two decades, the English statisticians "Student" and R. A. Fisher have been developing more exact methods of judging the reliability of conclusions, particularly where those conclusions involve correlation or are based on small samples. These new methods have as yet received but little recognition from American statisticians. They are presented here as simply as possible, and the discussion of the reliability of conclusions gives them full consideration.

So many persons have helped in the years during which this book has been growing that it is difficult for me to enumerate them all. First of all I should like to mention Howard R. Tolley, from whom I received my introduction to statistics, and with whom it has been a constant joy to work. I give him credit for much that is included here. The very order of presentation reflects that which he worked out for his classes. In a very real sense this book is a product of the spirit of research with which the Bureau of Agricultural Economics was imbued by the broad vision of Henry C. Taylor. John D. Black was the first to point out some of the undeveloped phases of statistical analysis, and then aided with encouragement and counsel in their solution. Bradford B. Smith aided in the beginning of the new developments, and his vivid imagination and logical mind have been a constant help. Among others who have collaborated in various stages, or who have independently worked out various phases of the problem, may be mentioned Sewall Wright, Donald Bruce, Fred Waugh, Louis Bean, and Andrew Court. Susie White, Helen L. Lee, and Della E. Merrick have given intelligent, conscientious, and loyal assistance in the clerical work in the development and testing of each new step.

In the preparation of the book itself I have had generous and willing help. Dorothea Kittredge and Bruce Mudgett have given the very substantial assistance of a detailed reading of the entire text, and many improvements in presentation and in material are due to their suggestions. For two terms the mimeographed manuscript has been used as a text in the United States Department of Agriculture Graduate School, and the members of the class have helped me in working out the illustrations, in clarifying the text, and in eliminating errors. R. G. Hainsworth, who prepared the figures, deserves credit for the excellence of the graphic illustrations. O. V. Wells helped in computing many of the illustrative problems, and Corrine F. Kyle in verifying the arithmetic. For the laborious and exacting work of typing the preliminary stencils, the many revisions, and the final manuscript, and for her care, patience, and suggestions, I am indebted to my mother, Rachel Brill Ezekiel; and for editing the manuscript and helping in the lengthy task of proofreading, to my wife, Lucille Finsterwald Ezekiel.

To all these, and to the many others who have helped me in the development of this work, I take this opportunity of expressing my obligation and my gratitude.

For any errors in the statements made and in the theories advanced, I alone am of course responsible. Although the text has been checked painstakingly, it is hardly to be hoped that a publication of this character will appear without some errors creeping in, in mathematics, in arithmetic, or in spelling. When such errors, or any ambiguities of statement, are noted by any reader, I would be very grateful if he would inform me of them.

MORDECAI EZEKIEL.

Washington, D. C.,
April 20, 1930.

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METHODS OF CORRELATION ANALYSIS

CHAPTER 1

MEASURING THE VARIABILITY OF A STATISTICAL SERIES

Statistical analysis is used where the thing to be studied can be reduced to or stated in terms of numbers. This is also true of many other forms of numerical analysis: surveying, physics, and chemistry employ numerical measurements, yet ordinarily they do not use statistical methods. That is because in those sciences it is usually possible to measure the particular thing or relation being studied so exactly that there is little or no doubt as to what is its true value. Statistical analysis, on the other hand, is used in the study of occurrences where the true value or relation cannot be definitely measured, or is hidden by other things, so that the numerical statement of the occurrence or of the relationship cannot be obtained directly from the original or "raw" figures. Instead, the data must be analyzed to determine the values desired.

The Arithmetic Average.—The simplest forms of statistical analysis have to do with averaging and classifying data. Thus if one were studying the yield of corn in one year in some area, say a county, for example, he might talk with 20 farmers picked at random and obtain figures, such as those in Table 1, showing the yield of corn which each farmer had obtained.

The most natural first step in reducing such a series of observations to more usable shape is to find the arithmetic average—to add all the yields reported and divide by the number of items. Performing this operation, the 20 reports total 600 bushels, or an average of 30 bushels.[1] This provides a single figure into which is condensed one characteristic of the whole group.

---

[1] Bushels are used here to represent any other quantity in which one might be interested in a particular case. If we let X represent the number of bushels reported by farmer 1, X' the bushels reported by farmer 2, X'' the bushels by farmer 3, and so on, we can then represent the sum of all the reports by the expression ΣX (read

TABLE 1

Yields of Corn Obtained by Twenty Farmers *

Farmer	Yield	Farmer	Yield	Farmer	Yield	Farmer	Yield
	Bushels per acre		*Bushels per acre*		*Bushels per acre*		*Bushels per acre*
1	29	6	33	11	29	16	33
2	25	7	26	12	35	17	31
3	38	8	28	13	26	18	37
4	30	9	30	14	23	19	28
5	27	10	29	15	31	20	32

* In making entries in a table such as this, the actual values may be "rounded off" to any desired extent. In this case they are rounded to the nearest whole bushel. For example, "33 bushels" represents any report of 32.5 bushels or more, and any up to but not including 33.5 bushels. If the original reports were secured to the nearest tenth bushel, this might be indicated by writing "32.5–33.4" instead of "33"; or if secured to the nearest hundredth bushel, by writing "32.50–33.49." The entry "32.5 to 33.5" will be used to indicate "from 32.5 *up to but not including* 33.5," whereas "32.5–33.4" will be used to mean "from 32.5 to 33.4, both inclusive."

But the average is not the only characteristic of the group which might be of interest. The average would still be 30 if every one of the 20 farmers had had a yield of 30 bushels per acre; yet there certainly would be a significant difference between 20 reports each of 30 bushels, and 20 reports ranging from 23 to 38 bushels, even though both did have the same average.

Classifying the Data.—One way of showing the differences in the individual reports is to arrange them in some regular order. If the farmers interviewed have simply been visited at random, and not selected so that those visited first represent one portion of the county, and those visited later another portion, the order in which the records stand has nothing to do with their meaning. As a first step to seeing

---

"summation of the X's"). Similarly, if we use n to represent the number of observations we have obtained, and use M_x to represent the *average* (or *mean*) number of bushels for all reports, we can define the *arithmetic mean* by the formula:

$$M_x = \frac{\Sigma X}{n} \tag{1}$$

This formula can be applied to anything we are studying, no matter whether X means bushels of corn, inches in height, degrees of temperature, or any other measurable quantity; or whether there are 2 cases or 2 million. This is a perfectly general formula which can be applied to any given problem. As statistics is a study of general methods, so stated that they can be applied to particular problems as desired, it will be necessary to use many general formulas of this sort. The student should therefore familiarize himself with the definitions given above, and the way they are used in formula (1), so that he will be able to understand and use each formula as it occurs.

just what the data do show they can be rearranged in order from smallest to largest, as shown in Table 2.

TABLE 2 ARRAY

Yields of Corn on 20 Farms, Arranged in Order of Increasing Yields

Bushels per acre

23	28	30	33
25	28	30	33
26	29	31	35
26	29	31	37
27	29	32	38

Looking at the series, it is now easier to tell something about the group of reports. One can now see that only one farmer had yields of less than 25 bushels per acre, and only two had more than 35, so that 17 out of the 20 had from 25 to 35, inclusive. Counting down the series shows, too, that ten of the farmers had less than 30 bushels of corn per acre, and ten had 30 or more, so that the figures 29 and 30 mark the middle of the number of yields reported. Dividing each half into halves again shows that five men had yields of 27 bushels or less, five had yields of 33 bushels or more, while ten men—half of those reporting—had yields from 28 to 32 bushels, inclusive. This tells something about how variable yields were from farm to farm in this area—half of the reports fell within this 5-bushel range.[2]

Even as rearranged in Table 2, the 20 reports still constitute a large tabulation. If there were several hundred, such a listing would be so unwieldy that it would be difficult to interpret.

Frequency Tables.—The records can be studied more easily if instead of writing ''29'' three times when there are 3 farmers with 29 bushels each, we simply show that 3 men each reported 29 bushels. Similarly, instead of putting ''30'' down twice, we can show that 30 bushels were reported by 2 men. Performing this operation for all the reports, the data can be assembled into what is known as a ''frequency table,'' which shows the frequency—that is, the number of times each yield of corn was reported.

In preparing a table such as Table 3, spaces are put in for all yields (such as 24 bushels) for which no reports were received, but

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[2] In statistical terminology, the figure which divides the number of reports into halves—as 29.5 in this case—is termed the *median*; and the figures which divide the numbers into quarters—as 27.5 and 32.5—are termed the *lower* and *upper quartiles*. The difference between the two quartiles, within which the central half of the reports fall, is termed the *inter-quartile range*.

TABLE 3

Frequency Table, Showing Number of Times Each Yield was Reported, by Individual Bushels

Yield of Corn	Number of times reported	Yield of Corn	Number of times reported
Bushels		*Bushels*	
23	1	31	2
24	0	32	1
25	1	33	2
26	2	34	0
27	1	35	1
28	2	36	0
29	3	37	1
30	2	38	1

which lie between the largest and the smallest report, to show clearly that no such yields were reported.

Table 3 is an improvement on Table 2, but it is still pretty long—and if the lowest yield had happened to be 15, say, and the highest 60, it would have been longer still. For that reason it is frequently desirable to group the reports, not only for a yield of a specified number of bushels but for yields within a certain range of bushels. Thus Table 4 is just the same as Table 3, except that instead of showing the number of reports by individual bushel groups, it shows the number of reports for groups covering 3 bushels.

TABLE 4

Frequency Table, Showing Number of Times Each Yield was Reported, by 3-Bushel Groups

Yield of corn	Number of times reported
Bushels	
22.5–25.4	2
25.5–28.4	5
28.5–31.4	7
31.5–34.4	3
34.5–37.4	2
37.5–40.4	1

The presentation is now condensed enough so that it can be readily understood. It is easy to see that most of the reports fell around 25.5 to 34.4 bushels, and that more fell near 30 bushels than anywhere else. Of course, the 3-bushel group is purely arbitrary, and any other convenient "class interval," as it is called in statistical terminology, could have been used. Thus, if a 5-bushel class interval had been selected, the convenient groups 19.5–24.4, 24.5–29.4, 29.5–34.4, and 34.5–39.4 bushels could have been established, giving frequencies of 1, 9, 7, and 3 for the four groups. Just what class interval makes the most satisfactory table for any given set of data depends upon how the data run and how much detail it is desired to show. Where convenient, class intervals of 10 or some fraction or multiple of 10 are most convenient—the example just given shows how much easier it is to comprehend the 5-bushel classes than the 3-bushel.[3]

Measures of Deviation

The average deviation.—Table 4 shows, in fairly compact form, the way that the several individual reports fall on each side of the average value. For some uses, however, it is desirable to have a single figure which expresses the ''scatteration'' of the whole group of reports, in just the same way that the arithmetic mean expresses the average yield of the whole group.

One way in which the tendency of the group to scatter either far from, or close to, the mean, may be measured, is by finding out how far, on the average, each report lies from the mean. The following tabulation illustrates the way in which this can be done:

TABLE 5

Computation of Average Deviation from the Mean

Original report	Mean	Report minus the mean
Bushels	*Bushels*	*Bushels*
29	30	−1
25	30	−5
38	30	8
30	30	0
27	30	−3
. . *		
Total.	. . . .	60†

* The remaining 15 reports are not shown in this table, though included in the total.
† The plus and minus signs are disregarded in making this total.

$$\text{Average deviation} = \frac{60 \text{ bushels}}{20} = 3 \text{ bushels}$$

---

[3] Where there is a tendency for the reports to be grouped around certain values, such as 5, 10, etc., it is desirable to take the class intervals so as to make these values fall in the middle of the groups. Thus, with a concentration on even 5's and 0's, the groups 2.5–7.4, 7.5–12.4, 12.5–17.4, etc., may be used.

In computing the average deviation, the plus and minus signs are disregarded in adding up the individual differences from the mean.[4]

The new figure, 3 bushels, is the *average deviation* of all the reports. It shows that the 20 individual reports differed from the mean yield of 30 bushels by an average of 3 bushels each. This furnishes a single figure which expresses how much or how little the individual yields differed from the average yield. If the group of 20 reports were being compared with another group of 20, all of 30 bushels each, the *average deviations* of the two sets would indicate at once the difference in their make-up, even though both sets had exactly the same average value of 30 bushels. The second set, with all the reports exactly equal to the average, would have an average deviation of 0, as compared to the 3-bushel average deviation for the first set.

Whereas the arithmetic average is a measure of the central tend-

---

[4] Before writing the general formula for the average deviation it is first necessary to have some way of writing *any* deviation. Using X to indicate any given report, as before, and M_x to indicate the arithmetic average of all such reports, the small x will be used to indicate the deviation of each report from the mean of all, thus:

$$\begin{aligned} X - M_x &= x \\ X' - M_x &= x' \\ X'' - M_x &= x'' \end{aligned} \tag{2}$$

and so on.

Similar to the previous usage, Σx (read "summation of all the small x's") is used to indicate the sum of the values such as x, x', x'', etc.

The average deviation, denoted by the sign δ, is then defined by the following equation:

delta

$$\delta = \frac{\Sigma x \text{ (taken without regard to sign)}}{n} \tag{3}$$

It is necessary to disregard the signs in taking this sum, as otherwise the sum would be zero. If the signs were not disregarded, the values added would be as follows:

$$\begin{aligned} &\text{For item 1, } x \ (= X - M_x) \\ &\text{item 2, } x' \ (= X' - M_x) \\ &\text{item 3, } x'' \ (= X'' - M_x) \end{aligned}$$

and so on to the last item

$$\text{item } n,\ x_n \ (= X_n - M_x)$$

So when the deviations were summed,

$$\Sigma x = \Sigma X - nM_x$$

but

$$M_x = \frac{\Sigma X}{n}, \text{ so } nM_x = \Sigma X$$

hence

$$\Sigma x = 0$$

ency of a group of reports, the average deviation is instead a measure of the "scatteration" of the individual reports—of their tendency to lie near to, or far from, the central value.

The standard deviation.—How far a group of reports tends to scatter from the mean of the group may also be measured by another coefficient which has certain advantages from a mathematical point of view. This measure is based on the deviation of each report from the mean, just as is the average deviation. After the individual deviations are computed, each one is then squared, these values are added together to give the sum, then divided by the number of items, and the square root extracted of this average of the squared deviations.

TABLE 6

COMPUTATION OF STANDARD DEVIATION FROM THE MEAN

Original report	Mean	Report minus the mean (= deviation)	Deviations squared
Bushels	*Bushels*	*Bushels*	*Bushels*
29	30	−1	1
25	30	−5	25
38	30	8	64
30	30	0	0
27	30	−3	9
. . *			
Total.	. . . .	. . . .	288

* The remaining 15 reports are not shown in this table, though included in the total.

The sum of the squared deviations, as shown in Table 6, is then divided by the number of items included in the group, and the square root of the result computed. The computation is as follows:

$$\frac{288}{20} = 14.4$$

$$\text{Standard deviation} = \sqrt{14.4} = 3.79 \text{ bushels}^{5}$$

---

[5] The Greek letter σ is used as the sign for the standard deviation. Using x to represent individual differences from the mean, as before, x^2 for the square of each of such deviations, and Σx^2 for the sum of all such values, the standard deviation is defined mathematically by the formula

$$\sigma_x = \sqrt{\frac{\Sigma x^2}{n}} \qquad (4)$$

Where the arithmetic average is a fraction, so that computing each individual

The new value, 3.79 bushels, is called the standard deviation. (It is sometimes called the root-mean-square deviation, because it is the square root of the mean of the squares of the individual deviations.)[6] In comparison to the average deviation, which was found to be 3 bushels, it is somewhat larger. That is a relation which always holds—the process of squaring the deviations tends to emphasize the largest deviations more than does merely averaging them together. With well-distributed observations, so that the distribution is "normal" or nearly "normal," the standard deviation is about one and a quarter times as large as the average deviation.[7]

deviation and squaring it would take much arithmetic for accurate work, the standard deviation may be computed more easily by the following formula:

$$\sigma_x = \sqrt{\frac{\Sigma(X^2)}{n} - (M_x)^2} \tag{5}$$

Here the original X values are squared instead of the deviations from mean, or x values. It can be readily demonstrated algebraically that the two formulas give identical values for σ_x.

Thus $\quad$ each $x = X - M_x$

each $x^2 = X^2 - 2XM_x + M_x^2$

hence

$$\Sigma x^2 = \Sigma X^2 - 2\Sigma X M_x + \Sigma M_x^2$$

but

$$\Sigma X = nM_x$$

and

$$\Sigma M_x^2 = nM_x^2$$

hence

$$\Sigma x^2 = \Sigma X^2 - 2nM_x^2 + nM_x^2$$

and

$$\Sigma x^2 = \Sigma X^2 - nM_x^2$$

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[6] For a shorter method of computing the standard deviation, when there is a large number of observations, see Note 1 at the end of this chapter.

[7] A " normal distribution " is such a one as will be obtained from a series of observations of a variable influenced only by a large number of random or chance causes, each one small in proportion to the total. Thus the values secured by tossing a number of dice, and noting the spots at each reading, tend to conform to a " normal curve." Variables composed of a large number of small, independent elements also tend to have a normal distribution. Since this distribution can be studied mathematically, it is possible to work out theoretically many of its properties. These theoretical characteristics of the normal curve are valuable in studying data where the distributions are nearly normal.

If the distribution of the original observations is fairly regular, as in Table 3, so that most of the reports come at about the middle values and then thin out to both ends (that is, if the distribution approximates normality), the standard deviation gives a measure of the range within which a definite proportion of the cases will be included. Specifically, if we take the range from the distance of the standard-deviation below the mean, to the distance of the standard-deviation above the mean, about 68 per cent of the records will be included. In this particular case the mean is 30.00 bushels, and the standard deviation is 3.79 bushels, so the range will be from 3.79 less than 30.00, or 26.21, to 3.79 more than 30.00, or 33.79. Comparing this with Table 3, we find that 13 farmers reported yields between 26.5 and 33.4 bushels, while 4 reported 26.4 or less, and 3 reported 33.5 or more. The range 26.5 to 33.4 thus included 13 out of the 20 cases, or 65 per cent, which comes close to the 68 per cent which would be expected for the range 26.21 to 33.79 provided the distribution of the data were normal.

For some uses, the square of the standard deviation has advantages over the standard deviation itself. Just as the standard deviation, 3.79 bushels in this case, may be thought of as measuring "variability," so the standard deviation squared, 14.4, may be thought of as measuring "average squared variability." The term "variance" has been suggested by R. A. Fisher, an eminent English statistician, to designate this squared variability, and that term will be used hereafter in this book when the standard deviation squared is to be referred to.

The relation of the three measures which have been discussed in this chapter—the mean, the average deviation, and the standard deviation—is illustrated graphically in Figure 1. Here the frequency distribution shown in Table 4 has been charted, showing the yield in bushels of corn along the bottom of the chart, and the number of reports falling in each group along the sides.[8]

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[8] Mathematically, the quantities which are measured from left to right, and shown along the bottom of the chart, as the bushels of corn are here, are called the "abscissas," whereas the quantities which are measured from bottom to top, and shown along the sides as the number of reports are here, are called the "ordinates." Since any point in the whole chart can be located by telling how far it is from the left side, and how high it is from the bottom, these two items tell exactly where any particular point in the figure should fall. Thus the line for the group from 28.5 to 31.5 bushels has for ordinate the height 7 farms, and the abscissas of the ends of the line are 28.5 and 31.5 bushels. The ordinate and abscissa, taken together, are called the "coordinates" of a point.

Besides showing the number of reports included in each 3-bushel group by the height of the continuous line, the position of the mean in about the center of the group of reports is indicated, and likewise the number of reports included within a range of both one average deviation and of one standard deviation on each side of the mean.

Summary.—This chapter has shown (1) how a series of measurements of any one variable, such as the yield of corn from farm to farm, may be classified into a frequency distribution which shows how the individual reports are distributed from high to low; (2) how an arithmetic average may be computed which shows the value around which all the reports center; and (3) how the variation of the individual reports from the average may be summarized by computing the average deviation or the standard deviation, either one of which serves as an indication of the variability of the items included in the particular series. Although these statistical constants, especially the arithmetic average, are frequently of value for themselves alone, they are discussed here because it is necessary to know how they are computed and what they mean before the next propositions to be discussed can be fully understood.

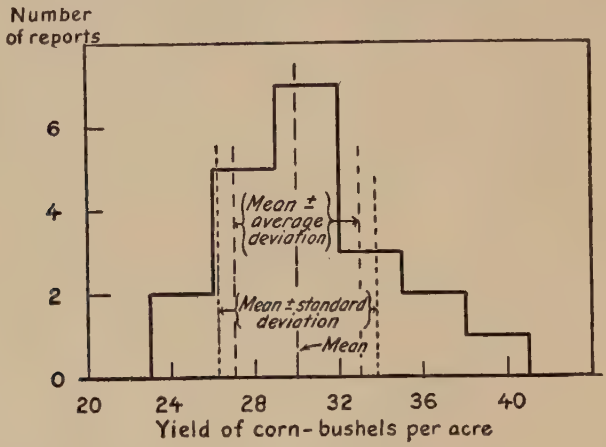

FIG. 1.—Frequency distribution of corn yields, and range above and below the mean included by average and standard deviations.

Note 1, Chapter 1.—Where the number of observations is large, the standard deviations may be computed more readily from a grouped frequency table than from the individual items. This process is illustrated in the following tabulation.

Yield	Number of reports (F)	Deviation from assumed mean (d)	Extensions	
			dF	d^2F
22.5 to 25.5	2	−2	−4	8
25.5 to 28.5	5	−1	−5	5
28.5 to 31.5	7	0	0	0
31.5 to 34.5	3	+1	3	3
34.5 to 37.5	2	+2	4	8
37.5 to 40.5	1	+3	3	9
Sums	20	...	+1	33

The standard deviation is then calculated from the grouped data by the formula

$$\sigma_u = \sqrt{\frac{\Sigma(d^2F)}{n} - \left[\frac{\Sigma(dF)}{n}\right]^2 - \frac{c}{12}} \qquad (6)$$

Substituting the values shown in the tabulation

$$\sigma_u = \sqrt{\frac{33}{20} - \left(\frac{1}{20}\right)^2 - \frac{1}{12}} = \sqrt{1.65 - (0.05)^2 - 0.0833} = 1.25$$

In making this computation, any convenient group may be selected as the assumed mean, and the deviations of the other groups (d) calculated as departures from it. This method assumes that all the cases in each group fall at the center of the group. With most variables, with a tendency toward a normal distribution, the average of the items in each group will fall somewhat nearer the center of the distribution than the mid-point of the group, so the use of this method tends to give too large a value for the standard deviation. The correction $-\frac{c}{12}$ called "Sheppard's correction" after its originator, makes an approximate allowance for this tendency. The c of the formula stands for the number of units of d in each class interval. Where a unit of 1 is used for each class interval, as in this problem, the correction becomes simply $-\frac{1}{12}$, to be applied to σ_u^2.

In computing the standard deviation from a grouped frequency table, the σ calculated will be in terms of the units in which d is expressed. In the illustration, each unit in d—one class interval—represents 3 units in X, since the yields were grouped in 3-bushel classes. The standard deviation computed in terms of class intervals, σ_u, is therefore only $\frac{1}{3}$ as large as is the standard deviation in terms of X. The latter may be calculated from the former by multiplying σ_u by the number of units in each group. That is,

$$\sigma_x = (\text{units of } X \text{ per class interval})\ \sigma_u$$

In this problem

$$\sigma_x = 3\ (1.25) = 3.75$$

The resulting value, 3.75, found by the short-cut method, is seen to be almost the same as the exact value of 3.79 bushels, previously found by the longer method. The greater the number of cases, and the more nearly normal the distribution, the more time will the short-cut method save, and the more nearly will its approximate result agree with the exact value found by the longer method.

CHAPTER 2

JUDGING THE RELIABILITY OF STATISTICAL RESULTS

Almost without exception, the object of a statistical study is to furnish a basis for generalization. In a case like that discussed in the preceding chapter, for example, no one would be apt to visit 20 farms scattered all over a county simply for the purpose of finding out what the yield of corn was on *those particular farms.* Instead, he might be studying the yield on those farms as a basis for determining what the average yield of corn was for all the farms in the county. Stated in statistical terms, he would be finding out what was the average yield in a *sample* of farms, picked at random, with a view to determining about what was the average yield in the *universe* in which he was interested, that is, on all the farms in the county.[1]

Of course it would be possible to visit all the farmers in the county, find out exactly what yield each one obtained, and so get an average of all the yields in the whole county. But this process would not only be expensive but in most cases would be a pure waste of time and energy. If the sample is made large enough, and the farms can be selected properly, the average yield can be determined to any desired degree of accuracy, and nothing is gained by getting reports from a larger number of farms. In this case, 100 records may enable one to determine the average yield quite as accurately as is necessary; obtaining records from all the several thousand farmers in the county might add nothing to the significance of the results.

Before considering ways of finding out how many records would be needed in any given case, it might be well to discuss a little more fully what the process of statistical inference involves. Really, all that we do is to examine or measure a certain group of objects, and

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[1] These two terms, "universe," meaning the whole group of cases about which one is interested in finding out certain facts, and "sample," meaning a certain number of those cases, picked at random or otherwise from all those in the particular universe, are both used frequently in statistical work, and should be clearly understood.

infer from the size or measurement of those objects, or from the way those objects behave, what will be the size of other objects of the same sort, or how other objects of the same kind will behave. This process is also called *induction,* because from particular facts about particular objects we lead out (*in duct*) *general* conclusions as to what will be the facts for all such objects in general. Now of course we do not really know what are the particular facts for any particular object without actually examining that individual object. All that we can do is to separate off certain groups of objects which we know to be like in one or more particulars, and then assume that they will tend to be like in other particulars too, even though we do not examine every one to prove it. In the case of our farms, all that we know about them is that they are in the same county. Now because they are in the same county, we may expect that the temperature will be about the same, the rainfall will be similar, and the growing season will probably not be much different from farm to farm. We may also expect that the kind of soil will not be very greatly different from farm to farm, and that the fertility will be somewhere near the same. Finally, we may expect that the fields are equally well-drained on the farms within the county.[2] But these expectations are not necessarily matters of known fact—we may expect that they are so from our general knowledge of the particular situation and of other similar situations. If the conditions agree with our expectations, generalizations from the facts of our sample to the facts of the universe as a whole may be quite correct; if conditions do not agree with expectations, then our general conclusions may be quite incorrect. In either case it is not a matter of statistical technique so much as a matter of logical interpretation. All that the statistical technique can do is to provide us with an average (or other measure or description of our facts) and a statement of how much confidence we can place in that average *under certain given assumptions.* Those assumptions may not be correct in any given case, and then our conclusion will be incorrect also; but that is not the fault of the statistics, but of the statistician; not of the facts, but of the use to which we try to put them.

Assumptions in sampling.—The basic assumptions upon which the theory of sampling rests apply both to the way in which the sample is obtained, and to the material which is being sampled. With respect to the material sampled, the assumption is that there is a large "universe" of uniform conditions, in that throughout the universe the individual items vary among themselves in response to the same causes and with about the same variability. With respect to the selection of sample, the values must be so selected (a) that there will not be any relation between the size of successive observations, that is, that the chances of a high observation being followed by another high observation will be just the same as of a low or a medium observation being followed by a high observation; (b) that the successive items in the sample are not definitely selected from different portions of the universe in regular order, but are simply picked at random so that the chance of the occurrence of any particular value is the same with each successive observation in the sample; and (c) that the sample is not picked all from one portion of the universe, so that another sample, from a different part of the universe, might be influenced by different conditions, but that the observations are scattered through the universe by purely chance selection.[3] Where these assumptions are fulfilled, the sample is said to be a "random sample," and its reliability may be estimated by the methods now to be described.

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[2] Obviously, these things would not be true in many sections. In hilly or mountainous areas temperature, rainfall, and length of growing season, may differ very greatly within short distances, whereas in other regions, such as the Coastal Plains areas, the soils may be so varied that very fertile and very infertile soils are jumbled together in a veritable crazy-quilt.

Condition

Taking up the question of how reliable a statistical average really is, the first thing which must be considered is, "What is the meaning of *reliable?*" If we are interested in corn yield, for example, it is obvious that a perfectly reliable sample would be one whose average agreed exactly with the average yield in the county. But if we were interested in knowing the average yield to within one bushel, then for that purpose the sample would be sufficiently reliable if its average came within one bushel of the average for the whole county.

Variations in successive samples.—Supposing that twenty farms had been visited at random, with the results already presented, if we wanted to find out how near the average from that sample hit the average for the county as a whole, we might try taking another sample—visiting twenty other farms at random, and getting the

Successi
Sample

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[3] Where the items are so selected as to represent different portions of the universe it may be called a "stratified sample"; where they are all selected from one portion of the universe, it may be called a "spot" sample.

Where the universe is not completely uniform, a "stratified" sample tends to be more reliable than a random sample, while a "spot" sample tends to be less reliable than a random sample. See G. U. Yule, Introduction to the Theory of Statistics, pp. 347 to 349 of 6th ed., for formulas as to the reliability of stratified and spot samples.

average yield for those 20. If the average yield of the second sample differed from the average of the first sample by, say, 3 bushels, we would know that both could not come within one bushel of the true average; if, however, the average of the second sample came within a half-bushel of the first average, we would be inclined to place more confidence in it. If we repeated the process several times over, and all the different samples had averages falling within one bushel of each other—say between 29.0 and 30.0 bushels—then we would feel pretty certain that the average yield for the county as a whole was 29.5 bushels, or very close to it.

Let us suppose that 15 more samples had been made, each from 20 farms scattered at random, and that when we tabulate the 16 averages from the 16 different samples, we have the following 16 values:

TABLE 7

Average Yield of Corn in One County. as Determined by 16 Different Samples of 20 Farms Each

Sample	Yield	Sample	Yield
	Bushels per acre		*Bushels per acre*
1	30.0	9	30.3
2	27.5	10	28.9
3	29.3	11	29.3
4	30.6	12	28.0
5	29.8	13	29.2
6	31.1	14	30.9
7	28.3	15	29.1
8	29.6	16	30.4

Although the 16 averages range all the way from 27.5 bushels for the smallest to 31.1 bushels for the largest, we can see that most of them fall around 29 or 30 bushels. This is even more evident when we arrange the 16 reports in a frequency table as shown in Table 8.

Although there is some tendency for the averages to cluster around 29 and 30 bushels, still there are several below 28.5 and several above 30.5. The average for the whole group is 29.5 bushels, and the standard deviation (computed as described on page 11) is 0.97 bushel, or, for practical purposes, 1 bushel.

Knowing that the standard deviation of the group of averages is 1 bushel tells us one thing about the way they scatter, from what we already know about the meaning of *standard deviation*—that about 68 per cent of them will fall in the range between one standard

TABLE 8

Frequency Table Showing the Number of Times Various Average Yields Were Obtained out of 16 Samples, by One-half Bushel Groups

Yield of corn	Number of averages in group	Yield of corn	Number of averages in group
Bushels		*Bushels*	
27.5–27.9	1	29.5–29.9	2
28.0–28.4	2	30.0–30.4	3
28.5–28.9	1	30.5–30.9	2
29.0–29.4	4	31.0–31.4	1

deviation below the mean and one standard deviation above the mean. In this particular case, the mean is 29.5 bushels, and the standard deviation is approximately 1 bushel, so the range of one standard deviation above and below the mean includes approximately from 28.5 bushels to 30.5 bushels. Checking this against the array of averages shown in Table 8, we find that this range does include 10 out of the 16 cases, or close to the proportion expected.

Now let us go back to our single original average of 30 bushels, based on visits to the original 20 farms. What we want to know is how reliable that one average is. Stated another way, how much is that average likely to be changed if the study were made over again—if another sample of the same size were taken?

In Tables 7 and 8 we have seen how it might actually work out if we *did* do the study over several times. We have seen that, in case the new averages did fall as shown in those tables, two-thirds of the new averages would fall within a range of 2 bushels. Furthermore, those figures showed that *all* of the different averages fell within a range of 4 bushels (from 27.5 to 31.5). But those conclusions were only obtained *after* getting 15 more samples of 20 cases each, and making 15 new averages, one for each sample. Is there any way to find out how much the single original is likely to vary from the true average without going to all the work of taking a number of new samples?

Estimating the Reliability of a Sample.—If we could estimate the extent to which the averages from new samples would be apt to vary, *without ever getting the new samples,* then we would know something more about how much faith we could put in the particular average which we had already. For example, if in the present case we knew that, if we did go out and get a large number of new averages (such as those shown in Tables 7 and 8), those new averages would have a standard deviation of one bushel, this fact would tell us at once *something* about how much our one average was likely to be different from the real average on all the farms. For example, we would know that about 68 per cent of the averages would lie in a range of 2 bushels (one standard deviation on each side of the mean). The one particular average which we had obtained might be any one of all those in a distribution like that shown in Table 8. If we assume that the mean of all the samples would coincide with the true average, then, as we have just seen, the chances would be about 68 out of 100 that our average was one of the averages falling within *one bushel* of the true mean. If, on the other hand, we knew that the standard deviation of a group of new averages would probably be, say, 5 bushels, then we would know that we only had about 2 chances out of 3 of coming within *five bushels* of the true average. Obviously, the average with 2 chances out of 3 of coming within one bushel of the true average would be much more reliable than the average with 2 chances out of 3 of coming within *five bushels* of the true average.

Whether we can judge how reliable a given average really is depends, therefore, on whether we can tell what would be the standard deviation of a number of similar averages, computed from random samples of the same number of items drawn from the same universe. If we could tell exactly what that standard deviation would be, we would know exactly how much faith we could put in the average we had—we would know just what the chances were of its being changed if the study were made over. Even if we did not know *exactly* what the standard deviation of the whole group of similar averages would be it would be some help if we knew approximately what it would be, or if we had a minimum or maximum value for its size, so that there would be some measure of how much trust to place in the particular average.

Computing the standard error.—Fortunately, it is possible to estimate what the standard deviation of a whole series of averages is likely to be, if each average is computed from a sample of the same size and drawn from the same universe.[4] This estimate is not precisely correct except under the exact assumed conditions, which are seldom completely obtained in practice. Even so, the ability to make a rough estimate is a tremendous aid to statistical investigators, for it affords some check on the accuracy of results, without going to the expense that would be involved in repeating every sample 15, 20, or more times, to make sure that a reliable result had been obtained.

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[4] Note 1 of Appendix 2 gives the derivation of this formula, and shows the specific assumptions on which it is based.

The method for estimating the standard deviation of the average uses just two values, both of which can be derived from the single sample from which the original average was computed. These values are simply the number of cases included in the sample, and the standard deviation of the individual items which made up that sample. First the square root of the number of cases (less 1) is taken, and then the standard deviation is divided by that square root, which gives the estimated standard deviation of the whole group of averages. Thus, for our original sample of 20 farms:[5]

Standard deviation of the average

$\frac{\sigma}{\sqrt{n-1}}$

$$= \frac{\text{stand. dev. of the cases in the original sample}}{\text{square root of (the number of cases in the sample} - 1)}$$

$$= \frac{3.79 \text{ bushels}}{\sqrt{19}}$$

$$= \frac{3.79 \text{ bushels}}{4.36}$$

$$= 0.87 \text{ bushels}$$

Allowing for a small number of observations.—Where there are only a small number of observations on which to base the computations, the standard deviation of the averages tends to be somewhat larger than the standard deviation estimated by means of equation (7), and hence the distribution of the averages from such small samples tends to be somewhat different from that indicated by the computed standard errors, according to the regular normal curve. If there are 30 or more observations, the difference is so small that it

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[5] Using the symbol σ for standard deviation as before, but using the subscript x to indicate the standard deviation of the individual items which go to make up our sample, and the subscript M to indicate the standard deviation of the means which is to be computed, thus:

σ_x = standard deviation of the items in the sample.

σ_M = estimated standard deviation of the group of averages if similar samples were repeated = *standard error* of the mean of X.

The standard error of the mean is then given by the formula

$$\sigma_M = \frac{\sigma_x}{\sqrt{n-1}} \tag{7}$$

Here, just as in the previous formulas, n stands for the number of items in the original sample—the same items as those from which σ_x was computed.

may be disregarded. The smaller the number of observations, however, the more serious the difference. A correction has therefore been worked out, by higher mathematics, to allow for this error in the estimated standard deviation where there are less than 30 observations, as shown in Table A.[6]

TABLE A

Probability that an Average (or Other Constant) Computed from a Sample Lies Farther from the True Value of the Constant than a Stated Number of Times the Computed Standard Error, for Varying Sizes of Sample. (Abbreviated Table.)*

For an error greater than the computed standard error multiplied by	Probability of such an error occurring, with specified number of observations in sample†						
	2	4	6	10	16	20	30 or more
0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
.50	.7048	.6514	.6382	.6290	.6244	.6228	.6171
1.00	.5000	.3910	.3632	.3434	.3332	.3298	.3173
1.50	.3744	.2306	.1940	.1678	.1544	.1500	.1336
2.00	.2952	.1394	.1020	.0766	.0640	.0600	.0455
2.50	.2422	.0878	.0544	.0338	.0246	.0218	.0124
3.00	.2048	.0576	.0300	.0150	.0090	.0074	.0027
3.50	.1772	.0394	.0172	.0068	.0032	.0024	.0005
4.00	.1560	.0280	.0104	.0032	.0012	.0008	
4.50	.1392	.0204	.0064	.0014			
5.00	.1256	.0154	.0042	.0008			

* See Figure A, Appendix 3, for full set of values.

† Based on article by "Student." New tables for testing the significance of observations. Metron V, No. 3, 105–120. 1925.

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[6] Table A applies as stated only in the case of constants such as the arithmetic average, which are computed from the original data by the determination of a single constant. Where the computation of the statistical constant involves simultaneously determining two constants from the original data, n-1 should be used for the "number of observations in the sample." This applies in the cases of coefficients of regression and of correlation. Where the computation of the statistical constant involves simultaneously determining a large number of constants, say j in number, from the original data, then $(n - j + 1)$ should be used for the "number of observations" in entering Table A or Figure A. Thus for a coefficient of partial regression, $b_{12.345}$ obtained from a sample of 20 observations, 5 constants are involved, so 16 would be used as the "number of observations" in using Table A to judge the reliability of the computed value. (Subsequent chapters will explain the meaning of the new coefficients mentioned here.)

The correction shows the probability of the observed average lying outside of the specified range from the true average. Thus if there are a large number of observations, there are 317 chances out of a thousand that the average from the sample differs from the true average by more than the computed standard deviation. For 20 observations, however, the corresponding chances are 330 out of a thousand. For a sample of only 2 observations, the chances of such an error increase to 500 out of 1,000.

Even with the adjustment from Table A, the estimated standard deviation of 0.87 bushel from our single sample of 20 cases, with an average of 30.0 bushels, tells us that with samples of that size, 67 per cent of the averages would probably fall within a range of 0.87 bushel of the true mean. If our sample is a true random sample, we therefore have 2 chances out of 3 of being within 0.87 bushel of the real average yield for all the farms in that section.

Interpretation of the Standard Error.—In comparison with the 15 other averages, all shown in Table 8, we see that in this case the standard deviation of all the averages was a trifle larger than we estimated it was likely to be—97 0.99 bushel, as compared to 0.87 bushel expected. It has already been noted that where a number of repeated samples are actually taken, this may easily occur. In practice, sampling rarely fulfills all the conditions on which the mathematical formula is based, and for that reason an average may be either less or more accurate than the computed standard deviation indicates that it is likely to be. Even so, this computed "standard deviation of similar averages" is an exceedingly useful figure. Such an estimated standard deviation for an average (or any other statistical constant) is called the *standard error* of that average (or other statistical constant). It serves as a standard measure to give warning by about how much that average may be in error for a specified probability, solely as the result of chance fluctuations in sampling, and hence indicates about how much confidence can be placed in that particular figure.

It is good statistical practice, whenever an average is cited, to give with that average its standard error, so that the reader will know about how significant that average is, and not be led into using it to make comparisons or to draw conclusions that are not justified by the number of observations which are summed up in that average. One way of doing this is to write the average followed by the statement "plus or minus the standard error." Thus, in the case we have been considering, with the single sample showing an average of 30.0

bushels with a standard error of 0.87 bushel, the correct statement is to say that the average yield has been shown by that sample to be 30.0 ± 0.87 bushels.[7] The reader then knows at once that the chances are about 2 out of 3 that the true average lies somewhere between 29.13 bushels and 30.87 bushels. If a similar sample from a different area had shown the average yield to be 28 ± 2.0 bushels, he would know that there was a fair chance that the true average yield was really the same in both areas, in spite of the difference shown by the two averages.

The greatest value of the standard error, however, does not lie in merely telling how close we may be to the true value *two times out of three.* In exactly the same way that we have seen that two-thirds (68.27 per cent) of the cases usually fall within *one* standard deviation on either side of the mean, mathematicians have determined for large samples that 19 out of 20 (95.45 per cent) will usually fall within *two* standard deviations of the mean, 369 out of 370 (99.73 per cent) will usually fall within *three* standard deviations of the mean, and all but one case out of 16,667 (99.994 per cent) will usually fall within *four* standard deviations of the mean. But where there are less than 30 observations in the sample, the tendency of the computed standard error to be misleading is even greater for these high odds than it is for lower odds. Corrections to take this into account are also shown in Table A. Thus, for a sample of 20 cases, there are 6 chances out of 100 of the sample giving an average differing from the true average by more than twice the computed standard deviation, and 7 chances out of 1,000 of its differing by more than three times the computed standard deviation. This last is three times as great as the probability of such a departure with an average based on over 30 observations. With a very small number of observations the differences for high odds are even greater. Thus, for 4 observations, there are 14 chances out of 100 of differing from the true mean by twice the computed standard error, and almost 6 chances

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[7] The most general practice is to write after the average ± .6745 times the standard error (0.59 bushel in this case, so the statement would read 30.0 ± 0.59 bushels). This value, 0.6745 σ_M, is called the *probable error* of the mean, since it gives the range within which the chances are even that the true mean lies, when there are more than 30 observations—and also the range *without* which the chances are even that the true mean lies. Since this tends to make the average appear rather more accurate than does the standard error, the practice suggested of using the standard error instead has been recommended by many competent statisticians. Wherever that is done, however, it would be well to insert a footnote explaining that it is the *standard error,* and not the *probable error,* which is being shown after the sign "±."

able
error

out of 100 of differing by three times the computed standard error. Where high reliability is desired, and only small samples are available, it is very important to take into account the corrections shown in Table A.

Our previous conclusion, based on the 20 observations shown in Table 1, that the true average yield in the area studied was "30.0 ± 0.87 bushels," may now be interpreted in any one of the following different ways:[8]

a. The true yield is between 29.13 bushels and 30.87 bushels, with one chance in 3 of being wrong. (1 time the standard error.)

b. The true yield is between 28.26 bushels and 31.74 bushels, with one chance in 17 of being wrong. (2 times the standard error.)

c. The true yield is between 27.39 and 32.61 bushels, with only one chance in 135 of being wrong. (3 times the standard error.)

d. The true yield is between 26.52 bushels and 33.48 bushels, with only one chance in 1,250 of being wrong. (4 times the standard error.)

Comparing these conclusions with the 16 samples shown in Tables 7 and 8, we see that 2 of those samples did fall outside the limits given by twice the theoretical standard deviation. If we had been so unlucky as to have gotten the worst one of these as our single sample, instead of the one which we actually did get, then we would not have hit the average even if we had used a range of twice the computed standard deviation as that within which we expected the true average to fall. On the other hand, every one of the averages fell within the range covered by three times the standard deviation. Even if, in picking our single sample, we had been unfortunate enough to draw the poorest one of the lot—the one which gave an average yield of 27.5 bushels—and had used a range of three times the standard error, we would have been correct as to our guess as to what the average really was. Then we would have concluded that the true mean fell somewhere between 24.3 and 30.7 bushels, which would have been wide enough to include the real mean. Of course, if we had taken four times the standard error, we would have been almost absolutely certain of being correct—there is only one chance in over 1,000 of being wrong.

In most statistical work, three times the standard error is taken as the greatest extent to which a given result is likely to be wrong.

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[8] Figure A, page 392, which gives in more detailed form the corrections shown in Table A, may be used to work out these odds.

Even though there is about one chance in 370 of being further off than this with samples of 30 or more, or one chance in 67, with samples of 10, most scientists are willing to take the chance that their sample is not that one exceptional case. For exceedingly important work, or where absolute accuracy of comparison is essential, even four times the standard error might be used; but for the general run of statistical problems, and with fair-sized samples, it would seem safe to regard three times the standard error as about the largest extent to which the conclusions might be out *solely because of the chances of getting* an *unusual sample* in random sampling.

The figure as to standard error tells nothing at all as to how much error there may be because of *bias* in sampling. Thus, if in taking our sample of 20 farms, we had visited only the largest farms with the most prosperous-looking buildings, we would be very likely to get a sample which was not representative of *all* the farmers in the county, but simply of the better ones, and so might get an average yield say 10 bushels above the true average for the county. Even if we only selected our farmers to the extent of including those which were most willing to give us the figures we wanted, we might have a badly biased sample, as usually it is the best farmers and the most intelligent ones who are most willing to answer such questions. We must depend largely on common sense, and not on statistical computations, to tell us whether or not our sample is really representative of the universe we want to study. All that the computed standard error can tell us is about how closely it measures the average (or other characteristic) *of the group it does actually represent*—whether that group is the one we meant it to represent or only a part of that group. This caution must always be kept in mind in using samples: computed standard errors tell us how far our results may be off solely because of the chance of getting a poor sample with a limited number of cases; but they do not tell us how far we may be off because of a *biased* sample, which is not a fair selection from the universe we wish to study.

Deciding on the size of sample necessary to obtain a stated reliability.—One other application of the standard error formula remains to be mentioned. The way in which this formula can be used to estimate the reliability of the average from a given sample, when the number of cases is known, has already been explained. The same formula can be used to determine how large a sample would have to be taken in order to secure results within any reasonable assigned limits of accuracy.

Thus it has already been shown that the records from 20 farms served to indicate that the true average yield lay somewhere between 27.39 and 32.61 bushels, with about one chance in 135 of its being outside of these limits. How many farms would one have to visit to determine the same average yield to within one bushel, with the same limit of accuracy? The same formula which was used to determine the standard error of the average can be turned around to answer this question also.

If we know that we want to get an average reliable to within one bushel, for a range of three times its standard error, then we know that the standard error of that average would have to be only one-third of a bushel. We may also assume that when we take our larger sample, the standard deviation of the yields on the individual farms will be found to be not very different from what it was in our sample of 20 cases, and so use the same standard deviation as we did before.

Taking the relation which was used in computing the standard error before:

$$\sigma_M = \frac{\sigma_x}{\sqrt{n - 1}}$$

In the new case we have the required standard error given, 1/3 bushel; we are assuming that the standard deviation of the cases in our larger sample will be 3.79 bushels, just as it was in our sample of 20 cases. Substituting these values in our equation, and using n'' to represent the number of cases required in the new sample, we then have

$$\tfrac{1}{3}\text{ bushel} = \frac{3.79\text{ bushels}}{\sqrt{n'' - 1}}$$

Shifting the terms around, this becomes

$$\sqrt{n'' - 1} = \frac{3.79\text{ bushels}}{\tfrac{1}{3}\text{ bushel}} = 11.37$$

Hence $n'' - 1 = 129.3$, and $n'' = 130.3$.

We therefore conclude that if a sample of 130 reports was obtained, we would get an average yield which we could be almost absolutely certain would not differ from the true average yield for all the farms by more than one bushel. If any other limit of error was set, we could similarly determine how many reports would be necessary to meet that limit.

Standard errors for other constants.—This whole discussion has been in terms of determining how closely it was possible to approximate the *true average* from the *average* shown by a sample. In exactly the same way standard error formulas have been worked out indicating how closely it is possible to approximate the true values of other statistical constants (such as standard deviations, for example) from the values for those constants determined from a sample.[9] These are interpreted in exactly the same way as are the standard errors of averages; they will be referred to in subsequent chapters.

Summary.—This chapter considers the question of how far statistical results derived from a selected "sample" drawn from a universe can be used to reach general conclusions as to the facts of the entire universe.

The confidence which can be placed in any constant computed from a sample, say an average, depends upon how closely that average is likely to come to the true average of the whole universe. One way of determining that would be to collect additional samples, each of the same size. From the way the averages from each of these different samples varied one could then judge how near the average from any one sample was likely to come to the true average. For samples which meet the conditions of simple sampling, another, and much more rapid, way is to compute the *standard error* of the average, which indicates the minimum extent to which the average is likely to be correct. With samples of over 30 cases, the true average will probably be within twice the standard error from the observed average for 19 samples out of 20, and within three times the standard error 369 times out of 370. This is the minimum error; where the number of observations is smaller, the possibility of error is larger, as is indicated by Table A, which shows the necessary corrections.

The same formula can be used to work out how large a sample must be taken to secure any desired degree of accuracy in the final average.

The estimated standard error does not take into account bias in selecting the sample, but only shows the chances of reaching incorrect results even when an honest random sample is obtained.

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[9] The standard error of a standard deviation (σ_σ) may be approximately determined by the formula

$$\sigma_\sigma = \frac{\sigma_x}{\sqrt{2(n-1)}}$$

It is subject to the same adjustments (Table A) as is the standard error of an average.

CHAPTER 3

THE RELATION BETWEEN TWO VARIABLES, AND THE IDEA OF FUNCTION

are Pearson

Relations are the fundamental stuff out of which all science is built. To say that a given piece of metal weighs so many pounds is to state a *relationship*. The weight simply means that there is a certain relationship between the pull of gravity on that piece of metal and the pull on another piece which has been named the "pound." We can tell what our "pound" is only by defining it in terms of still other units, or by comparing it to a master lump of metal carefully sheltered in the Bureau of Standards. If the pull is twice as great on the given piece of metal as it is on the standard pound, then we say that the lump weighs 2 pounds. If further, we say it weighs 2 pounds per cubic inch, that is stating a composite relationship, involving at the same time the arbitrary units which we use to measure extent or distance in space, and the units for measuring the gravitational force or attracting-power of the earth.

Relations between Variables.—Besides these very simple relationships which are implicit in all our statements of numerical description—weight, length, temperature,size, age, and so on—there are more complicated relationships where two or more variables are concerned. A variable is any numerical value which can assume varying or different values in different individual cases. The yield of corn on different farms is a variable, since it may differ widely from farm to farm. So is the length of time which a falling body takes to reach the earth, or the quantity of sugar that can be dissolved in a glass of water, or the distance it takes for an automobile to stop after the brakes are applied, or the quantity of milk that one cow will produce in a year, or the profit that a farm will pay in a year, or the length of time it takes a person to memorize a quotation. In contrast to these *variables* there are other numerical values called *constants,* because they never change. Thus one foot *always* contains 12 inches; one dollar *always* is equal to 100 cents; and a stone *always* falls 16 feet in the first second (under certain specified conditions).

Science, of any sort, ultimately deals with the relation between variable factors and with the determination, where possible, of the constants which describe exactly what those relationships are.

The variables which have been mentioned may be used to illustrate the way in which changes in one variable can be related to changes in another. Thus the length of time which a falling body takes to reach the earth varies with—that is, is related to—the distance through which the body has to fall. The quantity of sugar which can be dissolved in a glass of water varies both with the size of the glass and the temperature of the water. The distance it takes for an automobile to stop after the brakes are applied varies with the speed with which the car is traveling when the brakes are applied, the area of braking surface on the drums, the area of tire surface on the road, how tightly the brakes are applied, how much the car weighs, the kind of road, and so on.

Then when we come to variables like the production of milk or the income on a given farm, or the time to memorize a quotation, we find the situation still more complicated. How much milk a cow will produce varies with her age, breed, inherent ability, and the richness of the milk, and with the kind, quality, amount, and composition of the feed she receives, the way she is stabled and cared for, and many other similar factors. Similarly the variables which may affect the income on a farm—the size, the equipment, the crops grown, the livestock kept, the methods followed, the costs paid, the prices received, the rainfall—are so numerous that it would take an entire book merely to list and discuss the different factors affecting this one single variable. For the time it takes to memorize a quotation, its length, the subject's age, sex, training, fatigue or freshness, familiarity with the material discussed, and interest in the topic, all might affect the time required.

Yet it is precisely with relations between complex variables that many statistical studies must deal. The statistical methods which may be used to handle such problems can best be understood if presented first for the simplest cases, and then expanded to cover the more complicated ones.

Suppose a physicist, knowing nothing about the exact nature of the relation between the distance a body has to fall and the length of time it takes, made some experiments to determine the matter, and obtained the results shown in Table 9.

Looking over these figures we see that there is some sort of general relation between the two—as the distance increases, the time increases

TABLE 9

Relation Between Distance a Marble Drops and Time It Takes to Fall

Distance traveled	Time elapsed	Distance traveled	Time elapsed
Feet	*Seconds*	*Feet*	*Seconds*
5	0.6	20	1.1
5	0.5	20	1.1
5	0.6	20	1.2
10	0.9	20	1.1
10	0.8	25	1.2
10	0.7	25	1.3
15	1.0	25	1.2
15	0.9	25	1.3
15	1.0		

also. But that is not absolutely true—in one case the distance increased without there being any increase in the recorded time, while in some other cases the recorded time was not the same even though the distance was unchanged.

Graphic Representation of Relation between Two Variables.—We can get a better idea of just exactly what the relation is if we "plot" it on cross-section paper, so that we can see graphically just how the time does change with the distance. Figure 2 illustrates the way this is usually done. The units of one variable—in this case the distance to be traversed—are measured off from the left, starting with zero in the lower left-hand corner and counting over toward the right; and the units of the other variable—in this case the time elapsed—are measured off from the bottom, starting with zero at the bottom, and counting up toward the top. If negative values—less than zero

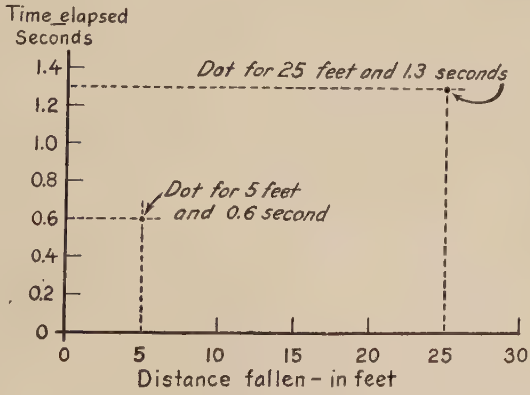

Fig. 2.—Method of constructing a dot-chart. Time elapsed is the dependent variable, and the distance is the independent variable.

—are present, then the counting is started with the *largest negative* value, decreasing from left to right or from bottom to top, until zero is reached and the positive values begin to appear.

Where one variable may be regarded as the cause, and the other variable as the result, it is customary to put the causal variable along the bottom. In this case it may be said that the differences in distance traversed cause the differences in time elapsed, so the distance is measured in the horizontal direction, and the time in the vertical. Of course there is no particular reason for plotting data just this way except that this is the customary way of doing it and so is most readily understood by other persons. Some relations of this sort can be reversed, so that either may be regarded as cause and either as effect.

Having laid off the chart in the way indicated, the next thing to do is to ''plot'' the individual observations. The way this is done is illustrated in Figure 2. The first observation was that it took 0.6 second for the marble to fall 5 feet. This is indicated on the chart by counting over to the 5-foot line from the left of the chart, and then counting up along that line until 0.6 second is reached. A dot is then placed on the chart at that point. As indicated, this dot is at the *intersection* of the line starting from the ''0.6 second'' at the left of the chart and extending parallel to the ''0 second'' line, with the other line starting from ''5 feet'' at the bottom of the chart, and extending parallel to the ''0 feet'' line. Similarly, the last observation, 25 feet in 1.3 seconds, is indicated by a dot where the horizontal line representing 1.3 seconds crosses the vertical line representing 25 feet.

Entering a dot for each individual observation in the same way, we get the chart shown in Figure 3. This figure now gives a visual representation of the way in which the length of time changes as the distance traversed changes. Such a chart is known as a ''dot chart'' or a ''scatter diagram.''

But even this figure does not show the *exact* relation between the distance and the time. Both the first and the second trials were for exactly the same distance, yet the time was slightly different. Obviously that difference in time could not have been due to the difference in distance between the two, because there was no difference. The investigator must therefore assume that some outside cause—perhaps the accuracy with which the time was measured—may have been responsible for these slight differences. It will be noted, too, that when the different observations are plotted as in Figure 3, they come close to all lying along a perfectly continuous curve; but we also see that the individual cases do not adhere absolutely to a continuous curve. If we are willing to assume, however, that all the differences between the different observations at the same point along the curve are due solely to extraneous factors, then we can estimate the true effect of the distance, by itself, by averaging together the several observations as to time taken for each of the several tests for the same length of fall. A continuous curve drawn through these averages would then indicate the way in which the duration of fall varied with the distance, *on the average* of the cases studied. Although it might not hold true for any one individual case, as we have just seen, still it does indicate *about* what the time will be. For practical purposes we may say that under given conditions the time a body takes to fall *is determined* by the distance which it has to fall.

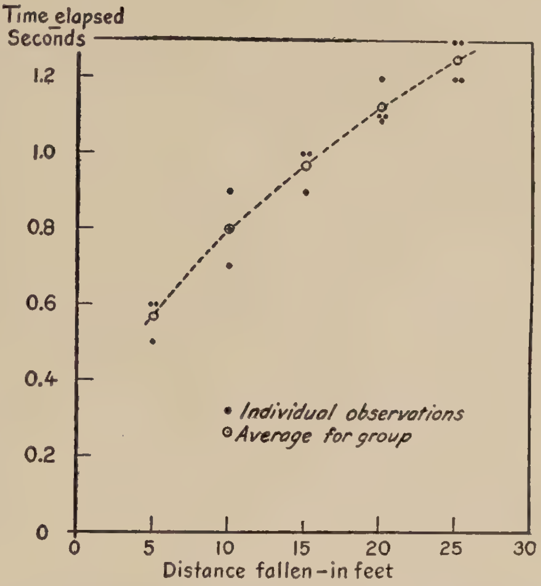

FIG. 3.—Relation of distance a marble falls to time elapsed in falling, as shown by individual observations and curve of average time.

The average time for each distance is indicated by the small circles in Figure 3. It is evident that these averages all lie very close to the smooth freehand curve which has been drawn on the chart.

Expressing a functional relation.—The relation shown by the curve in Figure 3 is what mathematicians call a *functional* relationship; the time it takes a body to fall *is a function* of the distance which it has to traverse.[1] All that this means is that for any particular distance-fallen, there is some corresponding time-required.

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[1] Using Y for time and X for distance, this is stated mathematically

$$Y = f(X)$$

The term "function" means that there is *some* definite relation between the two variables, number of feet and number of seconds, but it does not at all tell just *what* that relationship is. When, however, it is said that time is a function of distance according *to the curve shown in the figure,* then the statement has been made perfectly definite; the curve shows, for any given distance, exactly how long it will take a body to fall, on the average of a series of trials.

In this particular case the function is defined only by the graphic curve. It may also be stated as a mathematical expression

$= 16.1 t^2$ in a vacuum

$$Y = \tfrac{1}{4}\sqrt{X}$$

$ds = x$

$6t^2$

$= \frac{x}{16}$

$= \frac{1}{4}\sqrt{x}$

$= \frac{1}{4}\sqrt{x}$

using X for distance in feet and Y for time in seconds. This equation corresponds to the curve in a peculiar way, in that if any value of X is substituted in it, and then the value of Y determined, that will be the value of Y—time in seconds—corresponding to that particular value of X—distance in feet— as shown by the curve in Figure 3. This equation is therefore *the equation of the function,* since this simple mathematical expression tells just as much about the relation between the two varying quantities—time and distance—as does the entire curve in the figure.

The way this equation is used may be illustrated by two examples. Suppose a marble falls 16 feet, how long should it take to fall? The value of X would then be 16; substituting this value in the equation, we have

vacuum 16.1 ft.

$$Y = \tfrac{1}{4}\sqrt{16}$$
$$Y = \tfrac{1}{4}(4)$$
$$Y = 1.$$

This gives a value of 1 for Y, which means that it would take 1 second to fall. Suppose again a bomb were dropped from an airplane 10,000 feet high. How long would it take to reach earth? The value of X is then 10,000; substituting this value in the equation, it becomes

$$Y = \tfrac{1}{4}\sqrt{10{,}000}$$
$$Y = \tfrac{1}{4}\,(100)$$
$$Y = 25.$$

The result $Y = 25$ means that it would take 25 seconds for the bomb to fall.[2]

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[2] Outside causes, such as friction with the air, may make the time of fall slightly different from the calculated time; therefore with so long a fall as this the time might differ quite perceptibly from the theoretical time given by the equation. This equa-

It is evident that the equation even goes much further than does the graph of the curve. The latter gives the relation between distance and time only for the distances which are shown on the chart. The equation, on the other hand, gives the relation for any distance whatever, no matter what it may be. It is possible to state this *law of gravity,* as it is called, in an equation only because physicists have studied this relation in the past and determined exactly how the one quantity varies with the other. Having found that the same relation between the two variables held through their entire range of observation, and having worked out on philosophical grounds a good reason why that relation should hold, they have felt safe in coming to the conclusion that it will continue to hold even beyond the range of the experimental verification.[3] Where only a graph of the function is available, on the contrary, only the relation within the stated range is known; for the graph does not tell, of and by itself, the direction the curve would take if extended beyond the limits determined by the experiments.

Now if instead of the relation we have been just discussing we consider the relation between the quantity of sugar which can be dissolved in a glassful of water and the temperature of the water, we have a quite different problem, and yet one that is similar in many aspects. If we start in to determine it experimentally, we must first make sure that the quantity of water with which we are working is the same in every trial; then we must measure accurately both the temperature of the water and the amount of sugar which could be dissolved in it. Water expands when it is heated, and it also has a tendency to evaporate; so we would have to decide whether we wanted the *same volume* of water, irrespective of the fact that at a higher temperature there would be actually *less* water in that volume, or whether we wanted the *volume* of water equivalent to what would be the same volume at a given fixed temperature (which would necessitate determining the relation between volume and temperature, for a given weight of water as a preliminary study—or else using

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tion gives the time required *when no influence other than gravity* is taken into account. Obviously a marble would fall in air much faster than a feather—the resistance of the air has very little influence on the speed of the marble and a great deal of influence on the speed of the feather. In a vacuum they would fall at the same rate.

[3] It should be noted that for very great distances—say 10,000 miles—the formula might need to be modified, since then the pull of the earth would be less than it is at the surface. The equation holds true only for those distances from the earth within which its pull is practically a constant.

weight instead of volume as our criterion). Many other similar factors which might possibly influence the result would have to be considered before even the exact plan of the experiment could be drawn up.

But once the experiment had been run, the numerical results would probably be somewhat similar in character to those in the gravity test. It would be found that *about the same* quantity of sugar was dissolved in a given quantity of water when repeated tests were made at the same temperature, but that the quantities varied slightly from each other. If the data were plotted on a scatter diagram like Figure 3, it would be found that the data fell in the general shape of a curve, but that very few of the dots fell exactly on the curve, some lying above and some below the continuous line which could be drawn about through the center of them. Again we might conclude that these slight differences from exact agreement were due to factors other than the temperature of the water—to slight experimental errors in the quantity or temperature of the water, or to slight errors of measurement in determining the quantity of sugar—and be willing to conclude that the line drawn through the center of the series of observations showed the *real* effect of differences in temperature on the quantity of sugar dissolved, when extraneous influences were removed. This again would be a *functional* relation—the curve would express the relation between changes in temperature and changes in quantity of sugar, showing for any given temperature exactly how much sugar could be dissolved. It might then be possible to determine a type of equation which would accurately specify the function by a mathematical formula, similar to that discussed for the gravity example, if the logical type of relation between the two variables could be worked out.[4]

[4] Some logical foundation *is* needed before a mathematical equation to a curve can be of any more value than merely the chart which graphs the curve. Thus in the gravity example it is evident that the farther a body falls, the faster it falls; in every successive instant the speed it has already attained is increased by the effect of the continued pull which is added to it. Purely mathematical investigations of the relation between such constantly growing magnitudes and the variable with which they grow have enabled physicists to determine the general mathematical *type* to which the relation must conform; then, knowing what the type of the curve is, it is relatively easy to determine the constants (such as the "$\frac{1}{4}$" of the equation $Y=\frac{1}{4}\sqrt{X}$) which make the general equation applicable to a given specific case. This is done by using experimental results, such as those given in Table 9, to calculate the constants for the specific type of curve which has been determined upon.

Not all functional relations can be subjected to this type of logical analysis, however, and it is sometimes impossible to tell what sort of equation the results should really follow. In that case any mathematical curve "fitted" to the data has

Determining a Functional Relation Statistically.—In the two cases which have been discussed the relation between the two variables was sufficiently close so that by taking proper experimental precautions other influences which might affect the result could be largely removed, and a series of observations obtained sufficiently consistent with each other so that the exact nature of the relation could be readily determined. In many other types of relations this can not be done so easily—and it is with this type of relation that statistical methods really become important.

If we were making a traffic study in a given city, for example, we might wish to know what would be the safe speed limits to permit on different streets. In that connection we might need to know in what distance an automobile could be stopped when traveling at different speeds, so that by comparing this distance with the width of the different streets and the length of view at intersections we could judge how fast machines might be able to travel without risk of collisions at street intersections. One way to determine what is the relation between speed and stopping distance would be to make a number of tests in different portions of the city, taking different types of machines, and different drivers. Let us suppose that as the result of such a series of tests we obtained the series of observations shown in Table 10.

It is apparent from the table that there are great variations in the distances which different cars or different drivers required to stop, even when traveling at the same speed. This is shown even more clearly when we make a dot-chart of the data in just the same way as illustrated in Figure 3. The graphic comparison between speed and distance-to-stop, shown in Figure 4, reveals that there is only a general agreement between the different tests. There is certainly some relation between the two variables, but it is vague and uncertain in comparison with the relatively sharp and clear-cut relations shown in Figure 3.

There is no particular difficulty in understanding why the relation is not more definite. The data represent a great variety of different elements—cars with two-wheel brakes and cars with four-wheel brakes; cars with brakes in adjustment and cars with brakes well-worn; cars nearly empty and cars heavily loaded; cars with balloon

no more special meaning than the graphic curve drawn through the center of the observations; both are merely empirical descriptions of the relations, and both are limited in their interpretation to the range of the particular data upon which they are based. This fact will be discussed more fully later on.

TABLE 10

Relation Between Speed of Automobile and Distance to Stop After Signal, as Shown by 50 Individual Observations

Speed when signal is given	Distance traveled after signal before stopping*	Speed when signal is given	Distance traveled after signal before stopping*
Miles per hour	*Feet*	*Miles per hour*	*Feet*
4	2	19	46
7	4	24	93
17	50	14	26
14	36	12	28
12	20	9	10
11	28	10	34
20	48	15	20
15	54	24	70
17	40	25	85
13	34	20	64
15	26	19	36
19	68	13	26
10	26	10	18
18	56	7	22
22	66	16	40
18	84	14	60
8	16	20	52
4	10	24	120
12	14	24	92
20	56	17	32
23	54	13	34
18	76	11	17
12	24	13	46
16	32	14	80
18	42	20	32

* These observations were made before 4-wheel brakes were common.

tires and cars with high-pressure tires. In addition, the drivers differ —some are experienced drivers, some inexperienced; some strong and some unable to press the brakes fully down; some with almost instantaneous reaction to our signal to stop, some with faltering or lagging response; some bright and wide-awake, others tired and unobservant; some calm and steady, others nervous and erratic. Finally the conditions of the tests might be different—some on concrete pavement, others on asphalt; some on up-grades, some downhill.

There are two different ways by which we might go about deciding exactly what these varying observations showed. One way would be to divide up the data so that the effect of some of the different factors mentioned would be removed from the results. Thus if we separated the observations into different groups according to the make of car, and then reported each of these groups according to the model or the year made, the relation between speed and distance for any single group would be no longer affected by differences in braking equipment so far as engineering design went. Most of the remaining factors, however, would still be present to affect the results, so that even within each subdivision the records would still show great diversity in the relation. Only if we continued the process of subdivision of our sample until we got down to successive observations of a single car operated by a single driver at the same place, would we be apt to get observations as consistent with each other as those in the previous physical and chemical illustrations. Differences in the promptness with which the driver responded to the signal, in the preciseness with which the speed at the moment of giving the signal was observed, and possibly in the force with which the driver applied his brakes, might all influence the result, so that even then the results might be less consistent—"the curve be less definitely defined"—than in a series of laboratory experiments where *all* the important outside variables could be definitely controlled and so prevented from affecting the results obtained.

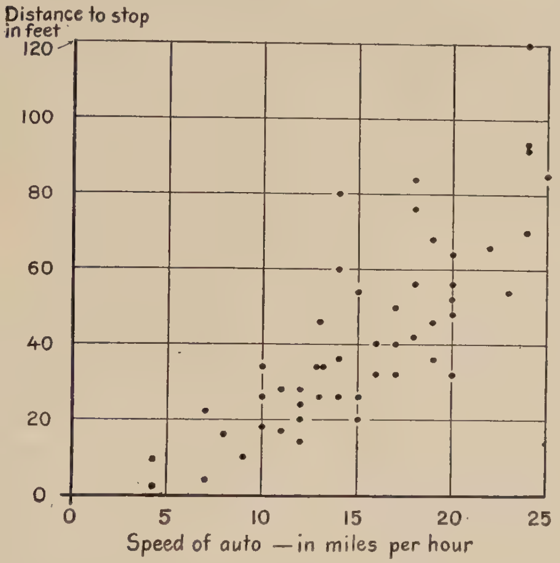

FIG. 4.—Relation of speed of automobile to distance it takes to stop, as shown by individual observations.

Should the entire mass of observations be analyzed as suggested, that would give a great number of different sets of relations, each one showing how long it took a given car to stop when driven by a given driver, when traveling at different speeds. But this great number of different curves might not be suitable to answer our question. They might be so different from curve to curve that it might seem that there was no real general relation between speed and distance—a new car, with four-wheel brakes, driven by an experienced driver, might stop in its own length at the same speed at which an old car, with brakes nearly worn out, and driven by an inexpert driver, might require a hundred feet or more. Obviously neither one of these extremes would be typical of the general relation; but what would be typical? Even the less extreme cases might show great variations among themselves, so that it would be almost impossible to pick from the great diversity of curves, one or a few that would serve as a basis of judgment for our problem.

A second way of going about it would be to try to determine some sort of *average* relation between speed and distance. In that case we would admit that there were great differences from the average in individual cases, yet would feel that the average would serve as a general indication of what the relation was, even though we were aware it would not be true in every, or perhaps even in any, individual case. If we knew *nothing* about a car except the speed at which it was moving, that average relation, however, would serve to give us the best guess we could make as to how far it would take it to stop. Since we would have to make our speed limits the same for all passenger cars, that might give us the best basis of judgment as to how high it was safe to place it. Of course we would also need to know something about how much *more* than the average time exceptional cars or drivers might require, and how far above the average any large proportion of them fell, so as to decide how much leeway to allow; but even so, the average relation would be the first interest and the point of departure in reaching our decision.

Where the relation between two variables is clear and reasonably sharply defined, as in the experimental case discussed, it is not difficult to determine the average relationship, since the relation for individual cases and the average relation for all cases, are nearly identical. Where the relation is not so well defined, however, and where many other relations are involved in addition to the particular relation which is being studied, it is by no means so easy to determine exactly what is the true relationship. A considerable body of statistical methods has therefore been developed to treat this particular problem. Since this problem pertains to the relation between variables, it has become known as the problem of co-relation, or ''correlation.'' Just how statistical technique may be applied to the solution of the traffic problem which has just been presented will be considered in detail in the next chapter.

Summary.—A statement of the change in one variable which accompanies specified changes in another is known as a statement of a *functional relation.* A functional relation may be stated either graphically by a curve or algebraically by a definite equation. Although functional relations may be readily determined from experimental conclusions where all influences except the one being studied are held constant, many problems cannot be studied by such methods. The statistical methods of *correlation analysis* may be used to study functional relations where experimental methods are not satisfactory.

CHAPTER 4

DETERMINING THE WAY ONE VARIABLE CHANGES WHEN ANOTHER CHANGES

(1) By the use of averages

The problem stated in the previous chapter, how many feet an automoble traveling at a given speed requires to stop, is to determine the *average* extent to which one variable changes when another variable changes. Stated mathematically, the problem is to find the functional relation between speed and distance—the probable distance required to stop with any given initial speed. Of the many different ways of doing this, the simplest, and the one which would suggest itself most naturally, would be to classify the records into groups, placing all of one speed in one group, all of another speed in another group, making as many groups as there are different rates of speed recorded, and then *averaging* the different distances for all the cases in each group. This would then give an average distance to stop for each given rate of speed in the series of records. Table 11 shows this operation carried out.

Where there were only single observations, this fact has been indicated by placing the average—the single report—in parentheses.

The averages in the last column of Table 11 show quite specifically how the distance required to stop tends to increase with the speed a machine is traveling. The machines which were tested at 12 miles per hour stopped at an average distance of 21.5 feet, those at 15 miles per hour at 33.3 feet on the average, and those at 20 miles per hour at 50 feet. But the increase is not uniform. The cars at 10 miles per hour averaged a greater distance than those at either 11 or 12, and the cars at 19, shorter stopping distances than those at 18.

If the successive averages from Table 11 are plotted and connected by lines, both the general increasing tendency, and the irregular change from group to group, are easily seen. Figure 5 shows this comparison.

Do these differences between the different group-averages have any real significance? Is there any reason to think that this very

TABLE 11

COMPUTATION OF AVERAGE DISTANCE TO STOP AFTER SIGNAL, FOR DIFFERENT INITIAL SPEEDS

Speed when signal is given	Different distances noted for that speed*	Average distance for that speed
Miles per hour	*Feet*	*Feet*
4	2, 10	6.0
7	4, 22	13.0
8	16	(16)
9	10	(10)
10	26, 34, 18	26.0
11	28, 17	22.5
12	20, 24, 28, 14	21.5
13	34, 26, 34, 46	35.0
14	36, 26, 60, 80	50.5
15	54, 26, 20	33.3
16	32, 40	36.0
17	50, 40, 32	40.7
18	56, 84, 76, 42	64.5
19	68, 46, 36	50.0
20	48, 56, 64, 52, 32	50.4
22	66	(66)
23	54	(54)
24	93, 70, 120, 92	93.75
25	85	(85)

* Data taken from Table 10.

jagged line is the *true* average relation between speed and distance? We can consider that from two points of view; the logic of the relation, and the statistical basis of the differences. Logically the differences are quite nonsensical. If a given machine can stop in 22 feet when it is going 11 miles an hour, of course it can stop in at least the same distance when going 10 miles per hour, probably something less, in fact—and it certainly would not take 26 feet, as the table shows. Then from the statistical point of view the groups are entirely too small to show very definitely how far on the average it takes to stop at *any* one speed. Even the largest group, at 20 miles per hour, has only 5 cases, while we have seen in Chapter 2 that 10 to 25 cases may be required as a minimum to give an average of much reliability. Computing the standard error for the average from the 20-mile group of reports, it comes out 5.3 feet, With only 5 reports, however, Figure A shows that we have to take a range of 1.1 times the standard error to make the chance of the true average falling within that range 2 out of 3, so we may say that the standard error of the average, taking this into account, is 5.83 feet. The average for this group of records may therefore be written 50.4 ± 5.8 feet; that is, there is at least one chance out of three that the average distance required to stop when traveling 20 miles per hour (for all automobiles in town, say) may be really less than 44.6 feet, or really more than 56.2 feet.[1] With the average from the *largest* groups showing as little reliability as this, it is quite clear that the zigzag variation from average to average has no real meaning—so few cases are included in each group that the averages are not statistically reliable to anything like the individual differences. All the irregular differences from group to group can therefore be accounted for by purely chance variations in sampling; it is quite possible that they are due solely to the small number of cases. As they have no statistical significance there is therefore no need to be worried about them.

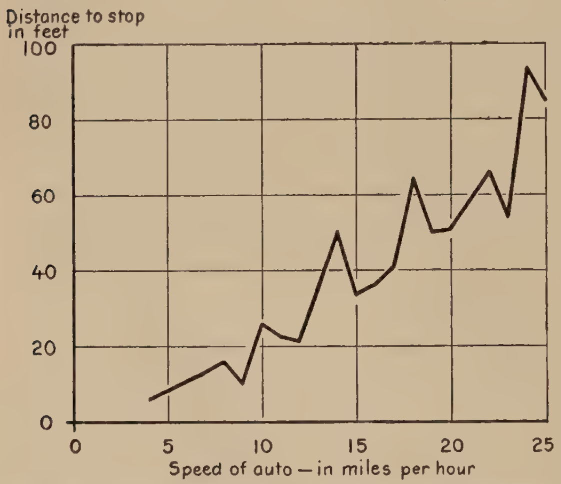

FIG. 5.—Relation of speed of automobile to distance it takes to stop, as shown by averages of small groups.

Does that mean that in spite of the relationship we can *see* in Figure 5 that we can get no accurate statistical measurement of the relation? That is overstating the case a little; all that we have determined so far is that the line of averages, the irregular function shown in Figure 5, has but little statistical meaning, *just as it stands now*.

We might be able to make the results more accurate by basing our averages on a larger number of cases—as we have seen previously, the more cases there are in a group the more reliable the average of that group is likely to be. One way of doing that would be to go out and get more records, so that we would have enough cases in each group to make the averages reliable within small enough limits to suit our needs. But that would be a long and expensive process—isn't there some way we can find out something more just from the records we have?

---

[1] The standard error is computed from the standard deviation of the five reports at 20 miles, using equation (7). The standard deviation of the five reports is 10.61; dividing by $\sqrt{n-1}$ gives 5.3. Figure A, in Appendix 3, shows that for 5 reports a range of 1.1 times the computed standard error must be taken to secure a reliability of 68, so the final standard error is (5.3) (1.1), or 5.83.

Another way of making the conclusions more stable would be by combining the records so as to give fewer groups, but with more cases in each group. So far we have been working with 19 different groups, one for each of the 19 different speeds measured. If instead we group them into a few groups—say four or five—we will have considerably larger groups to work with.

Independent and Dependent Variables.—The question might be asked whether the groups should be made on the basis of the rate of speed or of the distance to stop. (In preparing Table 11 we used the rate of speed without discussing the matter.) That comes back to the question of what we really want to find out. Do we want to know the *average speed at which machines were traveling* when it took them, say, 20 feet to stop; or do we want to know the *average distance* machines took to stop when they are traveling at a given speed? Obviously, the thing we are going to set is the speed limit, and we are merely interested in the distances to stop as one factor to guide us in deciding what the speed limit should be. We therefore want to know the effect of *speed* upon *average distance,* and not the reverse. For that reason we will classify our records on the basis of speed, and then average together all the different distances for the cars traveling at that speed.

The same question is met with in nearly all problems where the relation between two variables is to be dealt with. It is always necessary to think over the problem carefully, and decide which variable we are going to regard as the independent or *causal* variable, and which one as the dependent, or *resultant.* Thus if we were relating variations in tobacco yields to applications of fertilizer, obviously the differences in fertilizer would be the cause and the differences in yield the result, so we would sort our records according to the differences in fertilizer. Other relations may not be so clear-cut. If the size of stores were being related to profits, it might be as logical, in some situations, to consider that the more successful men were able to afford the largest stores, as to consider that the larger stores returned the greater profits. Careful consideration of the facts in each given case is necessary to clarify exactly what is the particular relation involved.

Larger Size Groups.—To return to our automobile problem. Since the speeds varied up to 25 miles per hour, and we have 50 reports to deal with, we might try breaking them up into five groups and see what kind of averages that will give us. Using groups covering a range of five miles per hour each, we can group the records and determine the averages for the five groups thus formed, getting the results shown in Table 12.

TABLE 12

Average Relation Between Speed of Car and Distance to Stop, as Shown by Records Thrown into Groups

Speed when signal is given *	Number of reports	Average speed	Average distance, to stop
Miles per hour		*Miles per hour*	*Feet*
Under 4.5	2	4.0	6.0
4.5 to 9.5	4	7.8	13.0
9.5 to 14.5	17	12.2	32.4
14.5 to 19.5	15	17.1	46.8
19.5 and over	12	22.2	69.3

* 4.5 to 9.5 means 4.5 and up to, but not including, 9.5.

These averages can then be plotted and connected by straight lines, just as were the averages in Figure 5. In constructing Figure 6, which shows this process, it is necessary to use the average speed as well as the average distance-to-stop in locating each point. This is because each of the average distances, as shown in Table 12, represents not one speed, but several different speeds thrown together; and if we wish to compare the average distances, it seems most sensible to compare them on the basis of the average of the speeds which they represent. The circles in Figure 6 represent the several group-averages plotted this way; the first one is located at the intersection of the lines for 4.0 miles per hour and 6.0 feet; the second at 7.8 miles per hour and 13.0 feet; and so on for the remainder.

When the group-averages of Figure 6 are connected by straight lines the relation between speed and distance is shown much more satisfactorily than it was in Figure 5. The line in the new figure shows a continuous relation between speed and distance. It indicates that when the averages are taken from groups large enough to eliminate the effect of individual cases, the higher the speed the greater the distance it takes to stop.

But on close examination even the relation shown in this last figure is not found fully satisfactory; for if we compute the change in distance to stop for each change of one mile in speed, we find that the conclusions are somewhat erratic. Thus between the first two averages —the change in speed from 4.0 to 7.8 miles per hour, an increase of 3.8 miles per hour, is accompanied by a change in distance from 6.0 to 13.0 feet, or an increase of 7.0 feet. Between 4 and 7.8 miles per hour, therefore, the distance to stop apparently increases 1.8 feet for each increase of 1 mile per hour in the speed of the machine. Similar computations for all the other groups are shown in Table 13, carrying out just the same process.

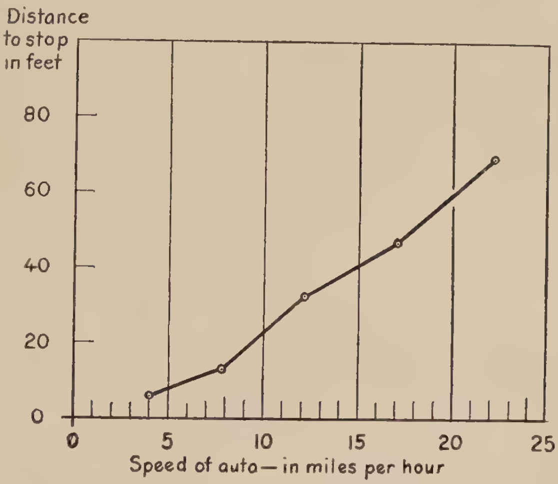

FIG. 6.—Relation of speed of automobile to distance it takes to stop, as shown by averages of large groups.

The results shown in Table 13 reveal that even the averages of Figure 6 are not altogether consistent. Between 4 and 8 miles per hour they indicate that the distance to stop increases 1.8 feet for each increase of one mile in the speed of the machine; between 8 and 12 miles per hour the distance suddenly starts increasing 4.4 feet for each one mile per hour increase in the speed of the machine; then between 12 and 17 miles per hour the effect of further increase on the speed becomes less again, averaging only 2.9 feet increase in a stopping distance for each increase of one mile per hour in speed, and then, finally, between 17 and 22 miles per hour changes again to 4.4 increase in feet to stop for each one mile increase in the speed of the auto.

This same variability in the rate of change can be seen directly from Figure 6 by noting the steepness of the several portions of the line. Between 4 and 8 miles per hour, where there is the least average change in distance for each change in speed, the line has the least slope, that is, is the nearest horizontal. Between 8 and 12 miles

TABLE 13

COMPUTATION OF CHANGE IN DISTANCE FOR EACH CHANGE OF ONE MILE IN SPEED, FOR DIFFERENT GROUPS OF RECORDS

Speed when signal is given	Average speed	Average distance to stop	Increase in speed	Increase in distance	Increase in distance per 1 mile increase in speed
Miles per hour	*Miles per hour*	*Feet*	*Miles per hour*	*Feet*	*Feet*
Under 5	4.0	6.0			
			3.8	7.0	1.8
5 to 10	7.8	13.0			
			4.4	19.4	4.4
10 to 15	12.2	32.4			
			4.9	14.4	2.9
15 to 20	17.1	46.8			
			5.1	22.5	4.4
20 to 25	22.2	69.3			

where the average distance to stop is much larger, the line tilts up abruptly; then between 12 and 17 miles per hour, where the average change in distance is less rapid, the line is flatter again, tilting up once more for the more rapid rate of change shown by the last group. It should be noted, too, that the slope of the line is almost exactly the same between the 7 and 12 mile averages, and the 17 and 22 mile averages, illustrating the fact that in both these intervals the increase in distance was the same for each mile per hour increase in speed. The irregular and zigzag character of the line in Figure 6 therefore shows the same vacillation in the group averages that the computations in Table 13 show, and simply by examining this chart closely it would have been possible to tell about this unsatisfactory character of the conclusions without taking the time to calculate out the exact rates.

But are the irregularities shown in Table 13 and Figure 6 of any significance statistically, or are they due simply to the possibilities of variation in using so small a sample, just as were the differences in Figure 5 and Table 11? Is it really true that an increase in speed has a larger effect upon the distance required to stop between 7 and 12 miles per hour than between 12 and 17?

Reliability of Group Averages.—The answer to these questions again involves a consideration of the statistical basis upon which our conclusions are based. These last results were calculated from the average speed and average distance for the several groups of records; obviously they can be no more reliable than are those averages themselves. In measuring the reliability of those averages by the methods we have already discussed, the thing to do is to compute the standard errors which will tell us about how much confidence we can have in each figure. That means that by calculating these statistical constants, we can judge at least the *range within which* it is probable that the true relation actually does fall.

The next step, therefore, is to calculate the standard error for each of the five averages of speed and the five averages of distance. The computation, which is exactly the same as that used before, based on equation (7), is shown in Table 14.

TABLE 14

Computation of Standard Errors for the Averages Shown in Table 12

Group	Number of cases, n	Standard deviation, σ	Computed standard error $\frac{\sigma}{\sqrt{n-1}}$	Range within which chances are $\frac{2}{3}$ that average will fall *	Average plus range for $\frac{2}{3}$ probability
	For speed				
Miles per hour		*Miles per hour*	*Miles per hour*	*Miles per hour*	*Miles per hour*
Under 5	2	0	....	....	4.0 ± ?
5 to 10	4	0.83	0.48	0.58	7.8 ± 0.6
10 to 15	17	1.39	0.35	0.36	12.2 ± 0.4
15 to 20	15	1.41	0.38	0.40	17.1 ± 0.4
20 and over	12	1.95	0.59	0.62	22.2 ± 0.6
	For distance				
		Feet to stop	*Feet to stop*	*Feet to stop*	*Feet to stop*
Under 5	2	4.00	4.00	7.84	6.0 ± 7.8
5 to 10	4	6.71	3.87	4.68	13.0 ± 4.7
10 to 15	17	16.09	4.02	4.18	32.4 ± 4.2
15 to 20	15	17.62	4.71	4.90	46.8 ± 4.9
20 and over	12	23.25	7.00	7.35	69.3 ± 7.4

* These values are obtained by adjusting the computed standard error to indicate the range for which the probability is only 0.33 that the true average lies outside. By interpolating in Figure A, the necessary adjustments to be applied to the computed standard errors are found to be: for 2 observations, times 1.96; for 4, times 1.21; for 15 or 17, times 1.04; and for 12, times 1.05.

Comparing the several averages with their respective adjusted standard errors, as shown in the last column of Table 14, we find that the average speed we have used for each group in constructing Figure 6 is quite reliable—there is not a great chance that, if we made the same number of observations over again, and used the same grouping, we would get averages different enough to change the appearance of the line. But with regard to the distance required to stop, the averages are much less reliable. If we collected enough records to determine the several averages quite accurately, there is one chance out of three that we might find that the true distance for the first group was practically nothing, or else more than 14 feet; or for the second group was less than 8 feet or more than 18 feet; and so on until for the last group it might be under 62 feet or over 77 feet. With this wide possible variation in the true values, it is quite evident that the real facts have not yet been measured accurately enough to justify detailed computations of the differences in the slope of different portions of the line. By changing any one of the averages as much as has been indicated, the slope of the line would be very materially changed.

Range within which true relation may fall.—The extent to which reliance may be placed in the relationship between the two variables as shown by the 50 observations which we have to deal with may be judged from Figure 7. Here the actual averages have been plotted, and lines drawn connecting them, just as before. But, in addition, rectangles have been drawn around each average to indicate the zone within which the true value would probably be found to lie if enough records were taken, using plus or minus the range for two chances out of three each way as the distance in laying off the rectangles from each average.[2] The corners of these rectangles have then been connected by lines just as were the averages before. The probabilities now are that the line showing the true average relationship between speed and distance would run somewhere between these upper and lower boundaries, even though it might not be the particular irregular line of averages we have used so far.

Figure 7 indicates that there is really a rather wide zone within which the true relation might fall. For example, it indicates that machines traveling 15 miles per hour would probably stop in 36 to 46 feet after the brakes were applied, whereas those traveling 20 miles an hour would probably stop in 52 to 68 feet. But this is still a pretty rough measure—would increasing the speed from 15 to 20 miles per hour increase the distance from 46 to 52 feet, only 6 feet; or would it increase it from 36 to 68 feet, 32 feet? Of and by themselves, the data do not tell us. We do not yet have any general statement of the relation between speed and distance.

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[2] As the rectangles have been laid off with regard to both distance and speed, there is only about one chance out of two that the true averages fall within them. Chances are two-thirds that the average for speed falls within the average speed ± the standard error; and then two-thirds that the average distance falls within ± its standard error, so they are only $\frac{2}{3}$ times $\frac{2}{3}$, or $\frac{4}{9}$, that both speed and distance would fall within their standard errors at the same time.

We have seen how increasing the number of cases included in a single group increased the dependence which would be placed in that group. However, even by reducing our 50 cases to 5 groups we have not been able to get a consistent and satisfactory statement of the relation. Is it possible that by handling all the data as a single group we could get a better result? One way of doing this would be to average all the speeds and all the distances together. But that would only tell us what was the average distance to stop and the average speed. What we want to know is what distance is most likely to be required at any given speed, and the treatment just suggested would not give us that.

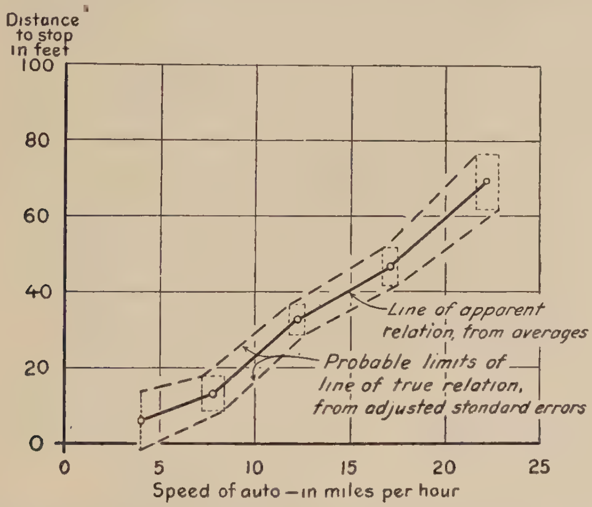

FIG. 7.—Relation of speed of automobile to distance it takes to stop, as indicated by the range around group averages for which the probability is $\frac{2}{3}$ that the true average is included.

There is one way, though, of determining the relation while considering all the records together. If we are willing to assume that an increase of one mile per hour in the rate of speed will increase the distance required to stop by exactly the same number of feet, no matter how rapidly or how slowly the machine is already moving, then we can determine this relation for all the data as a whole. On this basis a single line can be used to represent the relation and since the relation remains the same for all speeds, the line will have the same slope all along it; that is, it will be a continuous straight line. All that we have to do, therefore, is to determine a straight line which will come as near as possible to representing the relation as shown by all 50 individual observations.

Summary.—The change in one variable with changes in another may be approximately determined by grouping the records according to the independent variable and determining the corresponding averages for the dependent variable. Unless a very large number of observations is available, however, the functional relation shown by the successive averages will be irregular and inconsistent, due solely to sampling variability. For that reason some method is needed for measuring the functional relation for the group of records as a whole. The simplest way in which this can be done is by assuming that the relation can be represented by a continuous straight line. Methods of determining such a line will be considered in the next chapter.

CHAPTER 5

DETERMINING THE WAY ONE VARIABLE CHANGES WITH ANOTHER

(2) According to the straight line function

There are a good many ways by which a straight line can be determined to show the functional relation between the two variables, speed and distance. One way would be simply to place a ruler over the chart along the several group averages, or to stretch a black thread over them, and draw the line in by eye so as to fall as nearly as possible along them. While no two persons would draw their lines exactly the same, still this method might give fairly satisfactory results where only a rough measure was wanted. In the present case, however, in view of the expensive field work necessary to collect the data, it would seem worth while to put as much clerical time on analyzing those we have as is needed to give the most accurate results. We will therefore use the exact correlation method of determining the straight line.

The Equation of a Straight Line.—The determination of what this line will be consists in finding the *constants* for the *equation* of the line. Just as we have already seen (Chapter 3, page 32) that the curve showing the relation between the distance a body has to fall and the time it takes can be expressed by the relation,

$$Y = \tfrac{1}{4}\sqrt{X}$$

so any straight line can be expressed by the relation[1]

$$Y = a + bX \tag{8}$$

---

[1] Written this way, the equation is a perfectly general one which can be applied to the relation between *any* two variables, by calling one of them Y and the other one of them X. The symbol Y in the equation simply represents *the number of units* of the variable we designate as Y, whatever that may be, acres, dollars, pounds; and the symbol X likewise represents the number of units of the variable we designate as X. Thus if X is the number of rooms in each of a series of houses, X may be 4 for the first house, 7 for the next, 6 for the next, and so on. When we write X we

Figure 8 illustrates the meaning of a and b in this formula. When the value of X is 0, b times X is zero, and Y is equal to a. This constant, a, therefore, gives the height of the line (in terms of Y or vertical units) at the point where X is zero. This is indicated at the left edge of the chart.

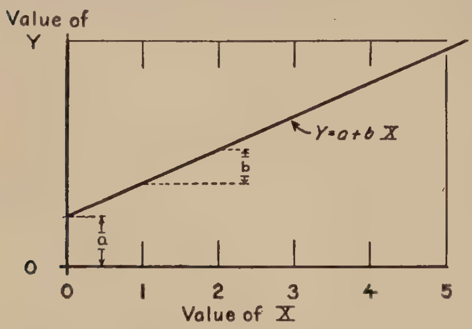

Fig. 8.—Graph of the function $Y = a + bX$

From the same equation, every time X increases one unit, Y increases b times one unit, since Y is computed as a plus b times X. The difference of the height of the line (measured in Y units) between the point where X is 1 and where X is 2, is therefore b units of Y, just as indicated on the chart. And this continues to hold true for every unit change in X, whether from 1 to 2, or from 0 to 1, or from 99 to 100.

The meaning of these constants in the *equation of the straight line,* as equation (8) is known, may be illustrated more concretely by taking some actual values for the constants a and b, and seeing how the line would look then. If we take 3 for a, and 2 for b, the equation would then read:

$$Y = 3 + 2X$$

Figure 9 shows the line for which this is the equation. Thus if X is taken as zero, the value of Y is found to be

$$Y = 3 + (2 \text{ times } 0) = 3 + 0 = 3$$

And 3 is therefore the Y-value corresponding to the X-value, zero.

Similarly if X is taken as 10,

$$Y = 3 + (2 \text{ times } 10) = 3 + 20 = 23$$

And the Y-value corresponding to the X-value of 10 is therefore 23. All other values of Y which may be computed for values of X within

then mean the number of rooms in each house, no matter how large or how small that number may be in any particular case. The particular number which X represents in any given case is said to be the value of X. Thus for a house of 5 rooms, we would say "the value of X is 5."

the range shown in Figure 9 will similarly be found to lie exactly on the same line.

Figure 9 illustrates again the meaning of the constants a and b. When X is zero, the value of Y is three units above zero, as indicated, and for every unit increase in X (say from 5 to 6) the value of Y goes up 2 units. This is exactly the same thing as shown in Figure 8, except that there no definite values were assigned to a and b, whereas here they have been given exact numerical values.

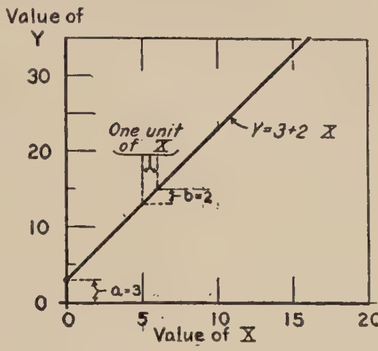

FIG. 9.—Graph of the function $Y = 3 + 2X$.

To represent the general relation between the speed of an automobile and the distance it takes to stop, therefore, we can use this same kind of equation, letting X stand for the speed in miles per hour, and Y stand for the distance to stop in feet.

Thus when we write the equation:

$$Y = a + bX$$

in this case we will be using that as shorthand for

$$\text{Feet to stop} = a + b\,(\text{speed in miles per hour})$$

But to give this equation definite meaning we must determine the numerical values for a and b, just as in our previous illustration we had to assume numerical values for these constants before the graph had any definite meaning for us.

The "observation equations."—One way of finding what the values should be is by regarding each one of our original observations (Table 10) as an algebraic equation itself. Thus the first observation, 2 feet to stop at 4 miles per hour, would be written

$$2 = a + b\,(4)$$

putting the 2 feet in place of Y in the equation, and the 4 miles in place of X.

Similarly the next observation, 4 feet to stop at 7 miles per hour, would be expressed

$$4 = a + b\,(7)$$

and so on right through to the last observation, 32 feet to stop at 20 miles per hour, which would be written—

$$32 = a + b\,(20)$$

Bringing all these different equations together would give a series looking like this:

$$\begin{aligned}
2 &= a + 4b \\
4 &= a + 7b \\
50 &= a + 17b \\
\cdot\ &\quad\ \cdot \quad\ \cdot\,\cdot \\
\cdot\ &\quad\ \cdot \quad\ \cdot\,\cdot \\
80 &= a + 14b \\
32 &= a + 20b
\end{aligned}$$

(The middle equations are omitted here to save space.)

Since we had 50 original observations, we would have 50 different equations, each one containing the two unknown constants a and b.

Now by the rules of simple algebra, any *two* independent equations containing *two* unknown constants can be solved simultaneously to obtain the numerical values for those constants. One way to find the values of our unknown a and b would be to pick two of the equations representing our observations and solve them simultaneously. Suppose we take the first and the last ones, we will then have:

$$\begin{aligned}
a + 4b &= 2 \\
a + 20b &= 32
\end{aligned}$$

Solving these two equations simultaneously, we find the values

$$a = -5\tfrac{1}{2}$$

$$b = 1\tfrac{7}{8}$$

But in getting these values we have used only 2 out of the 50 observations. Would we have gotten the same result if we had used another pair? Suppose we take the second observation and next to the last—

Then
$$a + 7b = 4$$

$$a + 14b = 80$$

These equations, solved simultaneously, give the values

$$a = -72$$

$$b = 10\tfrac{6}{7},$$

which are certainly far different from those secured before. Apparently the values secured by this method would depend upon the particular pair of observations selected, perhaps varying with each pair.

If we work out estimated values for Y for given values of X by these two solutions, we get estimates as follows:

According to the first result,

$$Y = -5.5 + 1.875X$$

When $X = 10, Y = 13.25$; when $X = 20, Y = 32$

According to the second result,

$$Y = -72 + 10.86X$$

When $X = 10, Y = 36.6$; when $X = 15, Y = 90.9$

If we should then plot the two calculated points for the first of these equations, and connect them by a straight line, we find that that line also passes through the two dots which represent the two observations from which the values were calculated. Similarly, if we should plot the two computed points for the second equation, and pass a straight line through them, that also would pass through the two dots which represent the values from which it was calculated. Clearly, therefore, fitting a line to two observations is merely determining the line that passes through them; and we could compute as many different lines as there are different pairs of observations *not* lying on the same line.

Fitting the Line by "Least Squares."—If we are going to use a mathematically-determined straight line at all, what we need is one which represents all 50 observations instead of any particular pair of them. No one line can *exactly* fit all 50 observations, for as we have just seen, the line which would agree with the first and the last would not agree at all with the second and next to the last. What we will have to find is some compromise line which will come as near as possible to agreeing with all the 50 observation equations, even though it does not *exactly* agree with any one. Mathematicians have worked out a method of obtaining such a line by the use of what is known as the "method of least squares." Although the process of determining the values of the constants a and b by this method is somewhat complicated, it takes all the observations into account, and gives each one of them an equal weight in the process. It is therefore of very great value in handling problems of this sort.

The equations upon which the process is based are derived by

TABLE 15

COMPUTATION OF VALUES FOR DETERMINATION OF LINE BY LEAST SQUARES

Speed in miles per hour, X	Distance to stop in feet, Y	X^2	XY
4	2	16	8
7	4	49	28
17	50	289	850
14	36	196	504
12	20	144	240
11	28	121	308
20	48	400	960
15	54	225	810
17	40	289	680
13	34	169	442
15	26	225	390
19	68	361	1292
10	26	100	260
18	56	324	1008
22	66	484	1452
18	84	324	1512
8	16	64	128
4	10	16	40
12	14	144	168
20	56	400	1120
23	54	529	1242
18	76	324	1368
12	24	144	288
16	32	256	512
18	42	324	756
19	46	361	874
24	93	576	2232
14	26	196	364
12	28	144	336
9	10	81	90
10	34	100	340
15	20	225	300
24	70	576	1680
25	85	625	2125
20	64	400	1280
19	36	361	684
13	26	169	338
10	18	100	180
7	22	49	154
16	40	256	640
14	60	196	840
20	52	400	1040
24	120	576	2880
24	92	576	2208
17	32	289	544
13	34	169	442
11	17	121	187
13	46	169	598
14	80	196	1120
20	32	400	640
Totals, $770 = \Sigma X$	$2{,}149 = \Sigma Y$	$13{,}228 = \Sigma(X^2)$	$38{,}482 = \Sigma(XY)$

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the use of calculus, and their derivation is given in the Technical Appendix, Note 2. The method itself, however, is very simple and can be used by anyone having a knowledge of simple algebra.

Computing the extensions.—The individual observations are first listed as shown in Table 15. The speed in miles per hour is placed under the heading "X," and the distance to stop in feet is placed under the heading "Y." Then each X item is squared, and entered in the column headed "X^2"; and each X item is multiplied by the accompanying Y item, and entered in the column headed "XY." Then all the items in each column are summed, giving the totals at the foot of each column. Just as before, in computing the standard deviation, we will use the symbols "ΣX" to represent the sum of all the X items; "ΣY" to represent the sum of all the Y items; "$\Sigma(X^2)$" to represent the sum of all the X^2 items; and similarly, we will use "$\Sigma(XY)$" to represent the sum of all the products in the XY column.

Solving the equations.—Having obtained these values as indicated in Table 15, we can next proceed to find the values of a and b by the aid of the following formulas:

$$b = \frac{\Sigma(XY) - nM_xM_y}{\Sigma(X^2) - n(M_x)^2} \tag{9}$$

$$a = M_y - bM_x \tag{10}$$

In using these formulas the value of b is determined first, then it is used in the next formula to determine the value of a.[2]

$$M_x = \frac{\Sigma X}{n} = \frac{770}{50} = 15.4$$

$$M_y = \frac{\Sigma Y}{n} = \frac{2149}{50} = 42.98.$$

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[2] It should be noted that if both X and Y had been stated in terms of deviation from their mean values (just as was done when the standard deviation, σ, was computed in Table 6), then they would have been denoted by the symbols small x and small y. If the product shown in the fourth column of Table 15 had then been obtained by multiplying together these two values, it would have been designated xy, and its sum, $\Sigma(xy)$. The correction factors used in the first part of the formula (9) just given, are used simply to change the product sum of the original observations, $\Sigma(XY)$, to what it would have been if it had been computed from the deviations of the mean instead. That is to say,

$$\Sigma(XY) - nM_x M_y = \Sigma(xy) \tag{11}$$

Similarly, $\Sigma(X^2) - n(M_x)^2 = \Sigma(x^2)$

Hence $b = \Sigma(xy)/\Sigma(x^2)$

Equations (9), (10), and (11) are only another way of stating the "normal equa-

Using the values for ΣX, ΣY, $\Sigma(X^2)$, and ΣXY given in Table 15, in equations (9) and (10), we find the values of b and a to be as follows:

$$b = \frac{\Sigma(XY) - nM_x M_y}{\Sigma(X^2) - n(M_x)^2} = \frac{38{,}482 - 50(15.4)(42.98)}{13{,}228 - 50(15.4)(15.4)} = \frac{5{,}387.4}{1{,}370} = 3.93$$

$$a = M_y - bM_x = 42.98 - (3.93)(15.4) = -17.54$$

The equation for the straight line, as thus determined by all the observations, is therefore

$$Y = -17.54 + 3.93\,X$$

This line is called the *line of best fit,* since it is the line which gives for all the 50 observed values of X, values of Y which come as near as possible to agreeing with *all* the different Y values observed. While some equations, such as the two computed from 2 observations each, would come closer than would this one for some individual cases, they would be much farther off for other cases; this one comes closer to agreeing with all the cases than any other straight line.[3]

Estimating Y from X.—We can see just how the equation for this line works by taking any given value for X we wish and working out what the estimated value for Y would be. That is, we can take any initial speed we wish and compute from the equation what would be the most probable distance required to stop, on the basis of the straight-line relationship.

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tions," which can be solved simultaneously to give the values for a and b. These equations are

$$na + \Sigma(X)b = \Sigma(Y)$$

$$\Sigma(X)a + \Sigma(X^2)b = \Sigma(XY)$$

These two equations can be solved simultaneously to get the values for a and b which will best fit all the equations, in the same way that the previous paired observations were put into simultaneous equations and solved simultaneously to get the values which would exactly fit the two observations.

The method by which this line is fitted rests upon the assumption that the scatter of the individual observations around the fitted line will approximate a normal distribution. If one or two observations are exceedingly erratic as compared to the others, so that the scatter of the observations around the line will be very skew, this method of fitting may be unsatisfactory.

[3] The way in which this equation gives the best fit may be explained mathematically. If the differences between each of the actual observations, and the estimated values given by this equation, are computed, squared, and summed, that sum will be smaller than it would be if any other straight line were used. Since this method determines the line with the smallest possible squared deviations, the line is known as the "least squares" line, and the method of computing it is known as the "method of least squares."

If 14 miles per hour is taken, then X will be 14. Substituting this value in the equation gives the estimated value of Y

$$\begin{aligned} Y &= -17.54 + 3.93\,(14) \\ &= -17.54 + 55.02 \\ &= 37.48 \end{aligned}$$

So the number of feet which would probably be required to stop, when traveling at 14 miles per hour, would be about 37.5 feet. Comparing this with the original observations, we see that the 4 cars recorded at this speed stopped in 36, 26, 60, and 80 feet, respectively. At 23 miles per hour the single car observed took 54 feet to stop. What estimate will the equation give for that speed? Let us see:

$$\begin{aligned} Y &= -17.54 + 3.93\,(23) \\ &= -17.54 + 90.39 \\ &= 72.85 \end{aligned}$$

This is much higher than the single observation. But referring to Figure 4 we see that that observation fell far below the general trend of the other observations. The straight-line equation, based on all the observations, thus seems to give a more reliable estimate of the distance which is most likely to be required to stop at any given speed than does any one individual observation.

But how far is it true that the straight line gives the most accurate estimate? Will it hold true for a speed of 1 mile per hour, or for a speed of 50? Let us see.

For 1 mile per hour the equation becomes:

$$\begin{aligned} Y &= -17.54 + 3.93\,(1) \\ &= -17.54 + 3.93 \\ &= -13.61 \end{aligned}$$

For 50 miles per hour it gives:

$$\begin{aligned} Y &= -17.54 + 3.93\,(50) \\ &= -17.54 + 196.5 \\ &= 178.96 \end{aligned}$$

Of these two results, only the latter sounds at all sensible. To say that a machine moving 1 mile per hour stops in *minus* 13.61 feet is saying that it stopped 13.61 feet *back* of where the brakes were applied, which is certainly nonsense. On the other hand, to say that a machine traveling 50 miles per hour would stop in about 179 feet after the brakes were applied might be quite reasonable—if we had any direct evidence for machines traveling at that speed. But that we do not have. All that we have are observations on 50 machines traveling at different rates from 4 to 25 miles per hour. Since we have no observations for speeds below 4 miles per hour, we can not expect our equation to be of any reliability below that point, and since we have no observations of speeds above 25 miles per hour, we cannot be sure that our equation will give good estimates beyond that point.

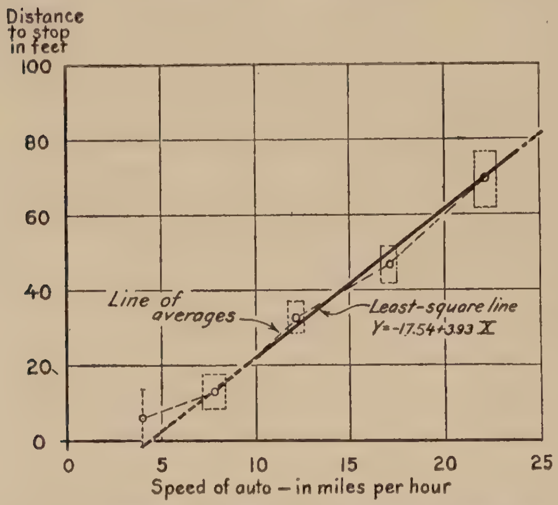

Fig. 10.—Relation of speed of automobile to distance to stop as indicated by ranges around group averages, and by least-square straight line.

Only within the range covered by the original observations can an estimating equation of this type be used.

Of the 50 observations, there were 6 below 10 miles per hour, and only one above 24, so 43 out of the 50 were between 10 and 24 miles per hour. For that reason no great reliance can be put in the equation below 10 miles per hour and above 24 miles per hour. Only within those limits where the bulk of the observations fell can the equation really be trusted.[4] For that reason the final equation, showing the average relation between speed and distance for automobiles, should be written as follows:

$$Y = -17.54 + 3.93\,(X), \text{ for values of } X \text{ between 10 and 24.}$$

Then the application of the equation is limited to the range given, and there is no danger of its being used to give absurd values for speeds too low, or untested values for speeds too high.

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[4] See pages 97 to 108 for a discussion of the type of problem in which a formula may be used to make estimates beyond the range covered by the data. See also Chapter 18 for formulas for estimating the standard errors for a and b.

Now that the limits of the line have been considered, it may be well to compare it to the group averages used before, to see how this single line, based on all the observations, compares with the irregular line obtained when the observations were grouped. This can be done conveniently by drawing in the line on Figure 7, which showed not only the line of averages but also the limits within which those averages were probably correct. This comparison is shown in Figure 10. The straight line determined by the least square solution has been drawn in solidly for the range of speed in which most of the observations fell, and has been dotted in for the balance of the range.[5]

Comparing the straight line with the group averages and the error limits within which they probably would fall, we see that the line does fall within those limits in every case but one, and in that case it just barely misses it. That shows that, so far as indicated by the number of observations we have on which to base the results, the straight line may give as reliable an indication of the distance in which a machine may be expected to stop as any of the individual group averages, and serve as a more reliable indication of the general relation than does the irregular line of the group averages.

The estimated distance required to stop, for each speed considered, is shown by the corresponding ordinate of the line in Figure 10. The exact estimated values may also be obtained by substituting the X value in the equation, just as has been done for the observations at 14 miles and at 23 miles. Carrying out this computation gives the estimated values shown in Table 16. Subtracting the estimated distances from the actual distances gives the *residuals,* or the difference between the two values. The symbol z is used in the table to designate these differences. The average of these differences, taken without regard to sign, is 11.6 feet; their standard deviation is 15.07 feet.

Interpreting the Linear Equation.—Just what does *the line of least squares* tell us, now that we have decided it is a fairly accurate indicator of stopping distances—at least within the range 10 to 24 miles? We can answer that by trying to explain what the constants

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[5] This line is drawn in according to the equation by determining the Y values for any two convenient values of X, and then drawing a straight line connecting them. Thus if the values at the end of the bulk of the observations, 10 and 24, are taken for X, the accompanying values for Y are found to be 21.8 and 76.8. These Y values are then plotted opposite 10 and 24 for X; a straight line drawn connecting them; and extended as a dotted line to cover the rest of the range.

TABLE 16

SPEED OF AUTO, DISTANCE TO STOP, AND DISTANCE ESTIMATED FROM SPEED BY LINEAR EQUATION

Miles per hour, X	Actual distance, Y	Estimated distance, Y'	Residual $(Y - Y')$, z	Miles per hour, X	Actual distance, Y	Estimated distance, Y'	Residual $(Y - Y')$, z
4	2	−1.8	3.8	19	46	57.1	−11.1
7	4	10.0	−6.0	24	93	76.8	16.2
17	50	49.3	0.7	14	26	37.5	−11.5
14	36	37.5	−1.5	12	28	29.6	−1.6
12	20	29.6	−9.6	9	10	17.8	−7.8
11	28	25.7	2.3	10	34	21.8	12.2
20	48	61.1	−13.1	15	20	41.4	−21.4
15	54	41.4	12.6	24	70	76.8	−6.8
17	40	49.3	−9.3	25	85	80.7	4.3
13	34	33.6	0.4	20	64	61.1	2.9
15	26	41.4	−15.4	19	36	57.1	−21.1
19	68	57.1	10.9	13	26	33.6	−7.6
10	26	21.8	4.2	10	18	21.8	−3.8
18	56	53.2	2.8	7	22	10.0	12.0
22	66	68.9	−2.9	16	40	45.3	−5.3
18	84	53.2	30.8	14	60	37.5	22.5
8	16	13.9	2.1	20	52	61.1	−9.1
4	10	−1.8	11.8	24	120	76.8	43.2
12	14	29.6	−15.6	24	92	76.8	15.2
20	56	61.1	−5.1	17	32	49.3	−17.3
23	54	72.9	−18.9	13	34	33.6	0.4
18	76	53.2	22.8	11	17	25.7	−8.7
12	24	29.6	−5.6	13	46	33.6	12.4
16	32	45.3	−13.3	14	80	37.5	42.5
18	42	53.2	−11.2	20	32	61.1	−29.1

a and b of the equation mean—the values −17.54 and 3.93, which we determined by least squares.

The first of these constants, a, is merely an empirical value to place the height of the line. If observations available and the type of equation used were such that they could be expected to give a sensible value for the distance to stop when X was zero—that is, when the machine was not moving—then a would give that value, since when $X = 0$, $Y = a$. But, of course, when a machine is not moving, it doesn't take it any distance to stop, so in this case the a has no sensible interpretation *at that point*. But that is to be expected—as has been seen, the line as a whole has but little meaning below ten miles per hour, and none at all below four miles; which was the lowest speed covered by the records. The constant *a*, therefore, has no meaning of and by itself in this particular example, but merely serves to place the height of the line as a whole for that range within which the line does have some meaning.

The constant *b*, on the other hand, is always significant. It shows the difference in *Y* for every difference of one unit in *X*, on the average of all the observations, and within the range covered. In this particular problem, the value of 3.93 for *b* indicates that between 4 and 24 miles per hour each increase of one unit in *X*, that is to say, each increase of one mile per hour in speed, causes on the average an increase of 3.93 units in *Y*—that is, of 3.93 feet in the distance required to stop. This interpretation of *b* can always be made, and is one of the most significant results secured by determining the constants for the straight line. In comparison with the values shown in Table 13, ranging from 1.8 feet to 4.4 feet increase in stopping distance for each one mile increase in speed, this figure of 3.93 feet per mile increase in speed is seen as a sort of weighted average, averaging together all the different possible sorts of comparisons like those in Table 13.

It should be noted that even though the straight line does fall within the standard error limits of most of the averages, as it does in this case, that that by itself is no proof that the straight line formula really expresses the true underlying relation as between the speed of a machine and the distance that it takes it to stop in this example. It is a purely arbitrary method of describing relation, which apparently expresses the observed relation fairly well; but that is all. It is, after all, only an empirical expression of the relationship; and because it happens to agree fairly well is no proof that it expresses the true nature of the relation. In fact, there is as yet no proof that it is even the best empirical description of the observed relation that can be obtained; further tests, to be described in the next chapter, are necessary.

But whether or not the straight line is the best function in this particular example, it is a type of relation of very great importance and usefulness. It is one of the simplest functions to fit and to explain, and for that reason it is very widely used. The equations used in determining the constants of the equation (equations 9 and 10, page 57) are therefore of great importance. The student of analytical statistics should become thoroughly familiar with the methods of determining the constants of the equation, and should understand thoroughly both the meaning and the limitations of this type of analysis.

Summary.—To express a functional relationship by a straight line, the constants may be determined arithmetically by the "method of least squares." Such a line gives the "line of best fit" under the assumptions of that method: a normal distribution of the observations around the line, and the reduction of the squared residuals to a minimum. Estimates of the dependent variable may be made according to the linear function for any value of the independent variable. Only within the range which includes the bulk of the independent values does this estimate have meaning, however; and only then if the straight line gives a satisfactory expression of the observed relation, either empirically or logically.

CHAPTER 6

DETERMINING THE WAY ONE VARIABLE CHANGES WHEN ANOTHER CHANGES

(3) For curvilinear functions

A straight-line equation is frequently a fairly good empirical statement of the relation between two variables even when the true relation is more complex than the straight line can portray. Yet it may be just as important to know the exact or approximate nature of the relationship as it is to have an empirical statement of it. For that reason it is necessary to consider other ways of expressing a relationship than the straight line.

In the automobile-stopping case we have been using as example, Figures 4 and 10 showed that the straight line agreed fairly well with the averages from the observations. Closer examination of the figures, however, reveals that for speeds below 10 miles per hour the actual stopping distance was usually greater than is indicated by the line; for speeds from 10 to about 17 miles per hour the average stopping distance was about the same as indicated by the line; above 20 miles per hour the stopping distance was frequently much greater than is indicated by the straight line. These considerations rob the line of much of its usefulness for the purpose for which the study was started—to serve as a basis for establishing speed limits. The linear relation between speed and stopping distance is apparently not accurate above 20 miles per hour, tending to underestimate the distance required at higher speeds. Since that might be the very range within which it was desired to set the speed, the conclusions most needed for that particular purpose would be lacking.

The real difficulty involved is in the assumption that the straight-line function applies. We have assumed that an increase of one mile in the speed of the car increases the distance required to stop by the same number of feet, no matter how fast the car is already traveling. When we examine Figures 5 and 10 closely, we see that this is not correct; the line of averages slants up slowly at first, then tends to rise more steeply as the speed is increased, until it has the steepest slope at the highest speed. It is therefore incorrect to assume that we can express the relation by determining the average increase in stopping distance for an increase of one mile in the rate of speed; for *the increase in stopping distance is not the same regardless of the rate of speed, but tends to become greater as the rate of speed increases.* Only if our expression of the relation can express that fact too will it sum up all our observations with sufficient accuracy.

What is needed is some general way of stating the relation between speed and distance, similar to the general relation expressed in the straight-line formula, yet expressing *a changing relationship* instead of the uniform linear relation shown by the straight line.

Different Types of Equations.—In the same way that it is possible to represent relations mathematically by a straight line, it is possible to represent them by curves of various types. We have seen how the equation $Y = a + bX$ can be used to represent any straight line by determining the proper values to be assigned to the constants a and b. There is practically no limit to the different kinds of curves which can be similarly described by mathematical equations. The equations of a number of curves which are useful in statistical analysis of the relations between variables, are as follows:

$$Y = a + bX + cX^2 \tag{a}$$

$$\log Y = a + bX \tag{b}$$

$$\log Y = a + b \log X \tag{c}$$

$$Y = a + b \log X \tag{d}$$

$$Y = \frac{1}{a + bX} \tag{e}$$

$$Y = a + bX + cX^2 + dX^3 \tag{f}$$

Each of these equations can be used to represent a certain type of curve. Thus type (a) is the equation of a parabola. If we take certain values for the unknown constants a, b, and c, substitute them in the formula, work out the values of Y for various values of X, and plot them the same as we did before, we will see the sort of curve this equation can be used to express. Thus if we take 1 for a, 0.5 for b, and $-$ 0.1 for c, the equation will read:

$$Y = 1 + 0.5X - 0.1X^2$$

When the value of X is 0, Y will be 1, obviously. When X is 1, Y will be

$$\begin{aligned} Y &= 1 + 0.5\,(1) - 0.1\,(1^2) \\ &= 1.4 \end{aligned}$$

When X is 2, Y will be

$$\begin{aligned} Y &= 1 + 0.5\,(2) - 0.1\,(2^2) \\ &= 1 + 1 - 0.4 \\ &= 1.6 \end{aligned}$$

Similarly, when X is 3

$$\begin{aligned} Y &= 1 + 0.5\,(3) - 0.1\,(3^2) \\ &= 1 + 1.5 - 0.9 \\ &= 1.6 \end{aligned}$$

For X equal to 4

$$\begin{aligned} Y &= 1 + 0.5\,(4) - 0.1\,(4^2) \\ &= 1.4 \end{aligned}$$

and for $X = 5$

$$\begin{aligned} Y &= 1 + 0.5\,(5) - 0.1\,(5^2) \\ &= 1 \end{aligned}$$

and for $X = 6$

$$\begin{aligned} Y &= 1 + 0.5\,(6) - 0.1\,(6^2) \\ &= 0.4 \end{aligned}$$

Plotting each of these values on cross-section paper and drawing a smooth curve through the several points, we get the result shown in Figure 11 in the center of the top section. Examination of the figures above and of this chart discloses one characteristic of this type of curve—the curve is always symmetrical on both sides of the highest point—the point where it stops going up and starts to turn down (as half-way between $X = 2$ and $X = 3$ in this case). The value of Y when $X = 2$ is the same as when $X = 3$. When $X = 1$ it is the same as when $X = 4$ and for $X = 5$, Y is the same as when $X = 0$. As a result the curve could be cut into two halves at the point of turning downward, one of which would be the reverse of the other. Besides this characteristic symmetry, this curve has another peculiarity—it has one, and only one, change from moving upward to moving downward, no matter what values are assigned to a, b, and c, or how far it is carried out. For the equation shown, the curve reaches its highest point when $X = 2.5$. As shown in Figure 11, the curve continues downward on both sides of this point, no matter how large the positive or negative values of X become. Thus if $X = 100$,

$$\begin{aligned} Y &= 1 + 0.5\,(100) - 0.1\,(100^2) \\ &= 1 + 50 - 1000 \\ &= -\,949 \end{aligned}$$

If $X = -\,100$

$$\begin{aligned} Y &= 1 + 0.5\,(-\,100) - 0.1\,(-\,100^2) \\ &= 1 - 50 - 1000 \\ &= -\,1049 \end{aligned}$$

Because of the characteristics mentioned, this type of curve is not very satisfactory to represent many types of relations. It does have great flexibility, in that many differently shaped curves can be represented by some particular segment of the parabola; but on the other hand the parabolic shape itself is so simple that many times the real relation between the variables can not be represented by a parabola.

The characteristics of a number of other types of simple curves are also illustrated in Figure 11. In each case an equation of the type indicated has been assumed, and the values of Y corresponding to values of X have been computed as has just been done for the simple parabola. Then plotting these computed values gives the curves shown. Thus type (f), the cubic parabola, is seen to have two points of maxima or minima and one point of inflection (the point where the curve changes from concave upward to concave downward, or *vice versa*). No matter what values are assigned the constants in this equation, it can have only the single inflection and the two points of maxima and minima. Of course the particular data to be represented might fall anywhere along the entire course of the curve—if only a single change from positive to negative slope were required, the point of inflection in the cubic parabola might lie beyond the extremes of the data, and so not show at all when the fitted curve was plotted for the range covered by the data.

Figures 11 also illustrates curves of types (b) to (e), as well as some others not given special type designations. In each case where the log of Y is used in place of Y, it is evident that the previous curve has been modified as if by compressing the ordinates nearest zero, and stretching out the ordinates farthest away from zero, stretching them more and more as they depart more and more from zero. This process transforms the straight lines of $Y = a + bX$ to a curve concave upward when log $Y = a + bX$ is used instead; or when log $Y = a + bX + cX^2$ is substituted for $Y = a + bX + cX^2$, it lengthens out the top of the bend if b is positive, or flattens out the bottom of the dip if b is negative. Similar results are found with the cubic parabola.

Similarly, when log X is used in place of X, the previous curves are modified as if the abscissas were compressed near zero, and stretched out in the higher values. This changes the straight line of $Y = a + bX$ to a curve for $Y = a + b$ log X, concave downward when b is positive, and concave upward when b is negative. The parabolas are similarly transformed, making the slopes different on each side of the bend in the simple parabola, or on each side of the inflection in the cubic. The effect is to move the "hump" or "dip"

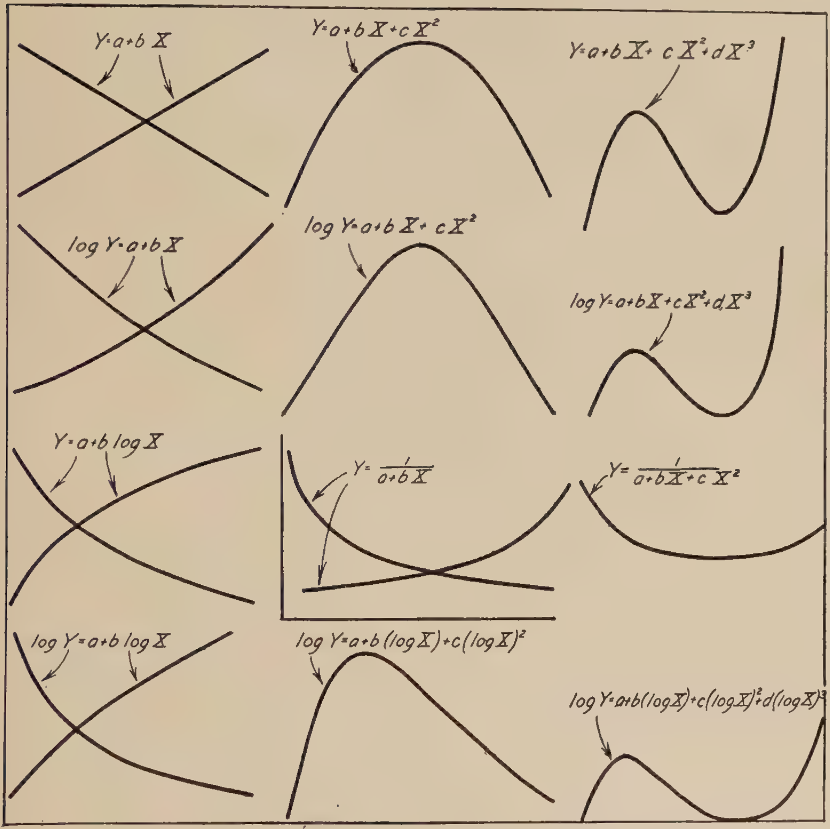

FIG. 11.—Curves illustrating a number of different types of mathematical functions.

in nearer to the zero abscissa, and to stretch out the remainder of the curve (including the second bend, in the case of the cubic parabola).

When logarithms are used for both X and Y, the effect is to modify both sets of coordinates in the manner previously described. The curve $\log Y = a + b\ (\log X)$ may have either a concave or convex bend if b is positive, but is always concave upward if b is negative. Similar modifications are noted in the case of the simple parabola.

In any case it should be noted that the curves whose equations contain logarithms retain some of the same characteristics as those with similar equations without logarithms. Thus the linear equations (with only a and b) *never* change from a positive to a negative slope; the simple parabola *always* has one such change, if carried out far enough; and the cubic parabola always has two such changes. In addition, it should be noted that a variable can be stated in terms of logarithms only if it has no negative values—whereas the other functions can express negative values as readily as positive ones, the logarithmic equations always become asymptotic as they approach zero—that is, they tend to flatten out and to run almost parallel with the zero axis. This is because a logarithm cannot be obtained for a negative number—no matter how small a logarithm becomes, the corresponding anti-logarithm is still positive, even if only a very small decimal fraction.

The hyperbola (type e) shown just below the center of Figure 11, also is peculiar in that it becomes asymptotic as it approaches both the X-axis and the Y-axis, even if one or both of the variables are in negative values. However, the values of X and Y which it approaches are not the zero values, as with the logarithmic curves, but special values which vary in each particular case, and which depend upon the value of the constants a and b in the equation. Still more complex curves of the same hyperbolic type may be obtained by including higher powers of X, such as

$$Y = \frac{1}{a + bX + cX^2}$$

The inability of any one equation to represent many simple curves may be illustrated by taking a different example than the automobile-stopping case we had been considering previously. Table 17 shows a series of observations of two variables—the protein content of different samples of wheat, as determined by chemical analysis, and the proportion of "hard, dark, vitreous kernels" in each sample, as determined by visual examination with the naked eye. The relation here is quite different from the one we have been considering so far. There is no causal connection between these two variables in the sense of one's being caused by the other. Instead, they are merely two different ways of measuring the character of the wheat. It is a short, rapid process, however, to examine the samples by eye, and determine the percentage of hard, dark, vitreous kernels, while it is a long and expensive process to run a chemical test on each lot. For that reason it is of importance to know whether it is possible to estimate the protein content from the percentage of vitreous kernels, and if so,

TABLE 17

Protein Content and Proportion of Vitreous Kernels for Each of a Number of Samples of Wheat*

Sample number	Protein content	Proportion of vitreous kernels
	Per cent	*Per cent*
1	10.3	6
2	12.2	75
3	14.5	87
4	11.1	55
5	10.9	34
6	18.1	98
7	14.0	91
8	10.8	45
9	11.4	51
10	11.0	17
11	10.2	36
12	17.0	97
13	13.8	74
14	10.1	24
15	14.4	85
16	15.8	96
17	15.6	92
18	15.0	94
19	13.3	84
20	19.0	99

* These values are actual items, picked so as to show the relationship more clearly. Actually, the correlation is not so high as is shown by these selected cases.

how closely. So even though the vitreous kernels do not *cause* the differences in protein, we can still regard the proportion of vitreous kernels as the independent variable; and regard the percentage of protein as the dependent variable. That means only that we are going to try to estimate the dependent (protein) from the independent (percentage of vitreous kernels) even though there is no direct cause-and-effect relation present.

The relation between the proportion of vitreous kernels and the per cent of protein may be seen more readily if a dot chart is made, showing the two variables for each of these individual observations. According to the previous discussion, we will regard the proportion of kernels vitreous as X, the independent variable; and the per cent of protein as the dependent variable, Y. In preparing the dot-chart, shown in Figure 12, we will therefore plot the X values, or per cent vitreous kernels, along the horizontal axis; and the Y values, the proportion of protein, along the vertical axis.

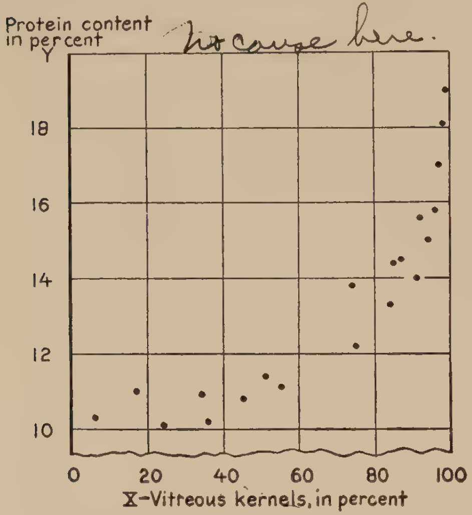

FIG. 12.—Dot-chart showing relation of proportion of vitreous kernels to protein content of wheat.

It is quite obvious from an inspection of the figure that a straight line would not do to represent the change in protein with change in vitreous kernels. Some type of curve is necessary. Let us see if the simple parabola is the proper type of curve.

"Fitting" a simple parabola.—To represent the relationship between the two variables according to the formula

$$Y = a + bX + c(X^2) \tag{12}$$

we will have to determine from the 20 observations the values to assign to the constants a, b, and c, just as before for the straight line we had to determine values for a and b. (Of course the a and b for the parabola will not be the same as the values for the straight line—unless c happens to be zero, which would make the equation for the parabola give a straight line instead.) The values for these constants are determined by constructing and solving the following equations:[1]

$$\left.\begin{aligned}(\Sigma x^2)b + (\Sigma xu)c &= \Sigma xy\\ (\Sigma xu)b + (\Sigma u^2)c &= \Sigma uy\end{aligned}\right\} \tag{13}$$

and

$$a = M_y - b(M_x) - c(M_u) \tag{14}$$

The values necessary in constructing equations (13) and (14) are derived as follows:

Use U to represent the X^2 values of equation (12)[2]

Then

$$\left.\begin{aligned}M_x &= \frac{\Sigma(X)}{n};\ M_u = \frac{\Sigma(U)}{n};\ M_y = \frac{\Sigma(Y)}{n}\\ \Sigma x^2 &= \Sigma(X^2) - n(M_x)^2\\ \Sigma xu &= \Sigma(XU) - n(M_x)(M_u)\\ \Sigma u^2 &= \Sigma(U^2) - n(M_u)^2\\ \Sigma xy &= \Sigma(XY) - n(M_x)(M_y)\\ \Sigma uy &= \Sigma(UY) - n(M_u)(M_y)\end{aligned}\right\} \tag{15}$$

After computing these values, the two equations (13) are solved simultaneously to obtain the values for b and c, and then these values are substituted in equation (14) to obtain the value for a.

Table 18, following, shows the form of computation in the first step to obtain these values for the data of Table 17.

The values at the foot of the table give the values called for in

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[1] An alternative method is to solve the following three equations simultaneously. The clerical work is about the same in both methods.

$$na + (\Sigma X)b + (\Sigma U)c = \Sigma Y$$

$$(\Sigma X)a + (\Sigma X^2)b + (\Sigma UX)c = \Sigma XY$$

$$(\Sigma U)c + (\Sigma UX)b + (\Sigma U^2)c = \Sigma XU$$

These equations are derived by the process explained in technical note 2.

[2] If U is made equal to X^2 divided by some convenient number, say 1000, the volume of necessary arithmetic can be materially reduced, without affecting the accuracy of the result. See technical note 3 for proof.

TABLE 18

Computation, for Wheat Problem, of Values Needed to Determine Constants of the Simple Parabola

x5625=421,875

Per cent vitreous kernels X	Per cent protein (minus 10)* Y	X^2 and U	XU	U^2	XY	UY
6	0.3	36	216	1,296	1.8	10.8
75	2.2	5,625	421,875	31,640,625	165.0	12,375.0
87	4.5	7,569	658,503	57,289,761	391.5	34,060.5
55	1.1	3,025	166,375	9,150,625	60.5	3,327.5
34	0.9	1,156	39,304	1,336,336	30.6	1,040.4
98	8.1	9,604	941,192	92,236,816	793.8	77,792.4
91	4.0	8,281	753,571	68,574,961	364.0	33,124.0
45	0.8	2,025	91,125	4,100,625	36.0	1,620.0
51	1.4	2,601	132,651	6,765,201	71.4	3,641.4
17	1.0	289	4,913	83,521	17.0	289.0
36	0.2	1,296	46,656	1,679,616	7.2	259.2
97	7.0	9,409	912,673	88,529,281	679.0	65,863.0
74	3.8	5,476	405,224	29,986,576	281.2	20,808.8
24	0.1	576	13,824	331,776	2.4	57.6
85	4.4	7,225	614,125	52,200,625	374.0	31,790.0
96	5.8	9,216	884,736	84,934,656	556.8	53,452.8
92	5.6	8,464	778,688	71,639,296	515.2	47,398.4
94	5.0	8,836	830,584	78,074,896	470.0	44,180.0
84	3.3	7,056	592,704	49,787,136	277.2	23,284.8
99	9.0	9,801	970,299	96,059,601	891.0	88,209.0
1,340	68.5	107,566	9,259,238	824,403,226	5,985.6	542,584.6

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* To simplify the following calculations, 10.0 has been subtracted from each protein reading. (See technical note 3.)

equations (15). Substituting the values as computed for those shown symbolically, the arithmetic appears as follows:

$$M_x = \frac{\Sigma(X)}{n} = \frac{1{,}340}{20} = 67$$

$$M_y = \frac{\Sigma(Y)}{n} = \frac{68.5}{20} = 3.425$$

$$M_u = \frac{\Sigma(U)}{n} = \frac{107{,}566}{20} = 5{,}378.3$$

$$\Sigma(X^2) - n(M_x)^2 = 107{,}566 - 20(67)^2 = 17{,}786$$

$$\Sigma(XU) - n(M_x)(M_u) = 9{,}259{,}238 - 20(67)(5{,}378.3) = 2{,}052{,}316$$

$$\Sigma(U^2) - n(M_u)^2 = 824{,}403{,}226 - 20(5{,}378.3)^2 = 245{,}881{,}008$$

$$\Sigma(XY) - n(M_x)(M_y) = 5{,}985.6 - 20(67)(3.425) = 1{,}396.1$$

$$\Sigma(UY) - n(M_u)(M_y) = 542{,}584.6 - 20(5{,}378.3)(3.425) = 174{,}171.05$$

These calculations give the values needed in equations (13), which are to be solved simultaneously to obtain the values of b and c. Substituting the values just computed in the equations gives the two equations to be solved as follows:

$$\left.\begin{aligned}(\Sigma x^2)b + (\Sigma xu)c &= \Sigma xy \\ (\Sigma xu)b + (\Sigma u^2)c &= (\Sigma uy)\end{aligned}\right\}\left\{\begin{aligned}17{,}786b + \quad 2{,}052{,}316c &= \quad 1{,}396.1 \quad (A) \\ 2{,}052{,}316b + 245{,}881{,}008c &= 174{,}171.05 \quad (B)\end{aligned}\right.$$

The simplest way to solve these is by the Doolittle method, as indicated in Appendix I, page 362.

Solving the equations simultaneously gives $b = -0.0879$, $c = 0.001442$. These values are then substituted in equation (14) to obtain the value for a.

$$\begin{aligned} a &= M_y - b(M_x) - c(M_u) \\ &= 3.425 - (-0.0879)(67) + (0.001442)(5{,}378.3) \\ &= +1.56. \end{aligned}$$

With our values for a, b, and c, we can now write out the equation for the parabola, $Y = a + bX + cX^2$ (12), for this particular case as follows:

$$Y = 1.56 - 0.088X + 0.00144X^2 \qquad \text{(I)}$$

Since 10 was subtracted from the per cent of protein before calculating the equation,[3] to estimate the actual per cent 10 must be added back in, making the equation read

$$Y = 11.56 - 0.088X + 0.00144X^2$$

This then is the equation of the simple parabola which comes nearest to describing the relationships between Y and X. From it the per cent of protein in a given sample of wheat may be estimated from the per cent of hard, dark, vitreous kernels in that sample.

We can see how the estimates are made by working them out for some of the samples. If we take the values of X for the first five samples in Table 16; 6, 75, 87, 55, and 34, for example, and substitute

[3] See technical note 3 for proof that this does not affect the values obtained for $\Sigma(x^2), \Sigma(xy)$, etc.

them in equation (I) above, we obtain estimated values for Y as follows:

When $X = 6$
$$Y = 11.56 - 0.088(6) + 0.00144(36) = 11.08$$

When $X = 75$
$$Y = 11.56 - 0.088(75) + 0.00144(5625) = 13.06$$

When $X = 87$
$$Y = 11.56 - 0.088(87) + 0.00144(7569) = 14.80$$

When $X = 55$
$$Y = 11.56 - 0.088(55) + 0.00144(3025) = 11.08$$

When $X = 34$
$$Y = 11.56 - 0.088(34) + 0.00144(1156) = 10.23.$$

TABLE 19

Comparison, for Wheat Problem of Actual Protein Content with Protein Content Estimated from Per Cent of Vitreous Kernels on Basis of the Simple Parabola

Per cent vitreous kernels, X	Per cent protein (minus 10), Y	Estimated per cent protein (minus 10), Y'	Difference between actual and estimated protein, $(Y - Y')$
6	0.3	1.08	−0.78
75	2.2	3.08	−0.88
87	4.5	4.83	−0.33
55	1.1	1.08	+0.02
34	0.9	0.24	+0.66
98	8.1	6.79	+1.31
91	4.0	5.50	−1.50
45	0.8	0.52	+0.28
51	1.4	0.83	+0.57
17	1.0	0.48	+0.52
36	0.2	0.26	−0.06
97	7.0	6.60	+0.40
74	3.8	2.95	+0.85
24	0.1	0.28	−0.18
85	4.4	4.51	−0.11
96	5.8	6.41	−0.61
92	5.6	5.68	−0.08
94	5.0	6.04	−1.04
84	3.3	4.35	−1.05
99	9.0	6.99	+2.01

Substituting each of the values of X in the formula in turn in a similar manner, we obtain estimated values for Y as shown in Table 19. So as to distinguish between the actual values of Y, and the values for Y estimated from X according to the equation of the parabola, we will designate the latter as Y' values.

It is quite apparent from the table that the actual and the estimated values generally fall rather near each other, the estimates part of the time being too high, and part of the time too low. We can get a better idea of the relation between the estimated and actual values by plotting both on a dot chart (Figure 13), similar to the way we did in Figure 12, using dots as before to represent the values of Y originally observed, and crosses to represent the estimated values, Y'. Since the Y' values are all computed from the formula, the crosses all lie on a continuous smooth curve, which we can sketch in free-hand, as indicated by the dotted line in the figure. Now if we want to estimate the protein for a sample with a proportion of vitreous kernels not included in our problem, say 65 for example, we can determine it either by substituting 65 for X in equation (I), and computing it out, or by reading from our smooth curve the Y value corresponding to an X value of 65. Of course this *graphic determination,* as it is called, will not be quite so exact as will the actual computation, but for most practical purposes the result will be quite sufficiently accurate.

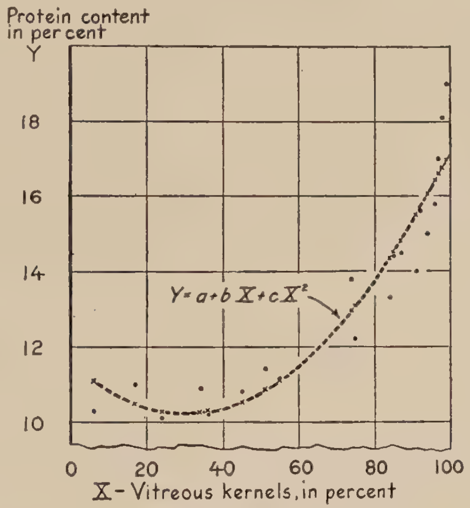

FIG. 13.—Dot-chart showing relation of vitreous kernels to protein content of wheat, and parabolic curve fitted to same.

Let us now examine Figure 13 and decide whether the formula for the parabola gives a satisfactory "fit" in this case—whether the estimated values do agree fairly well with the actual. We see at once that the curved line of the estimates does come closer to agreeing with the actual values than any straight line could. But on the other hand we see that the general shape of the parabolic curve and the general trend of the actual relationship is rather different. For low proportions of vitreous kernels, the estimated values are generally too low; for the highest proportions, they are also generally too low, while for proportions of vitreous kernels ranging from 70 to 95 per cent the estimates are too high.

Apparently the equation of the simple parabola is not adequate to describe this particular relationship. Especially for high proportions of vitreous kernels, the estimates are quite inaccurate. For 99 per cent vitreous, the parabola would estimate 17.0 per cent protein, while both samples over 97 per cent vitreous kernels had over 18 per cent protein. The failure of this curve to give a satisfactory "fit" is not due to any error in the computations, but merely to the fact that this formula cannot give the proper shaped curve to fit the relationship in this case. The mathematical properties of the equation itself are such that, no matter what constants are used for a, b, and c, it cannot come any closer to describing the true relation. The method just used in computing a, b, and c gives the *best* values for this case; any other three values substituted in the same formula would do even less well in "fitting" this particular set of observations.

"Fitting" a Cubic Parabola.—The cubic parabola, type (f) of the equations on page 66, might be tried to see if it would describe this particular relationship more closely.

The equation of the cubic parabola,

$$Y = a + bX + cX^2 + dX^3 \tag{16}$$

has four constants a, b, c, and d to be computed. Here again, of course, a, b, and c will be different from those we have computed previously, unless the d value comes out zero. The values b, c, and d are computed by the simultaneous solution of the following three equations:[4]

Use U to represent the X^2 of equation (16), and V to represent the X^3.

$$\left.\begin{aligned}(\Sigma x^2)b + (\Sigma xu)c + (\Sigma xv)d &= \Sigma(xy)\\ (\Sigma xu)b + (\Sigma u^2)c + (\Sigma uv)d &= \Sigma(uy)\\ (\Sigma xv)b + (\Sigma uv)c + (\Sigma v^2)d &= \Sigma(vy)\end{aligned}\right\} \tag{17}$$

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[4] The alternative method here involves the simultaneous solution of 4 equations, as follows:

$$\begin{aligned}na + (\Sigma X)b + (\Sigma U)c + (\Sigma V)d &= \Sigma Y\\ (\Sigma X)a + (\Sigma X^2)b + (\Sigma XU)c + (\Sigma XV)d &= \Sigma XY\\ (\Sigma U)a + \Sigma(UX)b + (\Sigma U^2)c + (\Sigma UV)d &= \Sigma UY\\ (\Sigma V)a + (\Sigma VX)b + (\Sigma UV)c + (\Sigma V^2)d &= \Sigma VY\end{aligned}$$

The value for a is then computed from the following equation:

$$a = M_y - b(M_x) - c(M_u) - d(M_v) \tag{18}$$

The values for Σx^2, Σxu, Σxy, Σu^2, and Σuy are computed as shown previously, equations (15). The additional values required in equation (17) are computed as follows:

$$\left.\begin{aligned} M_v &= \frac{\Sigma(V)}{n} \\ (\Sigma uv) &= \Sigma(UV) - n(M_u)(M_v) \\ (\Sigma xv) &= \Sigma(XV) - n(M_x)(M_v) \\ (\Sigma v^2) &= \Sigma(V^2) - n(M_v)^2 \\ \Sigma(vy) &= \Sigma(VY) - n(M_v M_y). \end{aligned}\right\} \tag{19}$$

It should be noted that among the values required to "fit" this cubic parabola, that is, to determine the constants a, b, c, and d, are the values such as $\Sigma(V^2)$ and (ΣUV). Remembering that $V = X^3$, and $U = X^2$, this means calculating X^5 and X^6. For $X = 10$, $X^6 = 1{,}000{,}000$, so for values of X such as those in Table 17, ranging from 6 to 99, it would take a tremendous volume of computation to compute the values required in equations (17), (18), and (19). This may be reduced by letting $U = X^2/100$, and $V = X^3/10{,}000$. The computation is not shown here in detail. It follows the general form of that given in Table 17; and the solution of the equations (17), starting in just as shown on page 167, may be most conveniently carried through by the method shown subsequently on page 362.

Even when the cubic parabola is "fitted" to the data given, however, it does not give a satisfactory "fit." Thus Figure 14 shows the cubic parabola fitted to the data, worked out as just described. The values found gave the equation

$$Y = 0.35 + 0.0345X - 0.1397(X^2/100) + 0.1788(X^3/10{,}000)$$

or, clearing of fractions,[5]

$$Y = 0.35 + 0.0345(X) - 0.0014(X^2) + 0.000018(X^3)$$

Adding in the 10 which was subtracted from Y before making the computations, the equation becomes

$$Y = 10.35 + 0.0345X - 0.0014X^2 + 0.000018X^3$$

[5] See technical note 3 for proof of this step.

In Figure 14, the original observations are represented by dots, the estimated values from the cubic parabola are represented by stars, and the curve of the simple parabola is also shown. A curve has been drawn through the stars to show the general shape of the cubic parabola.

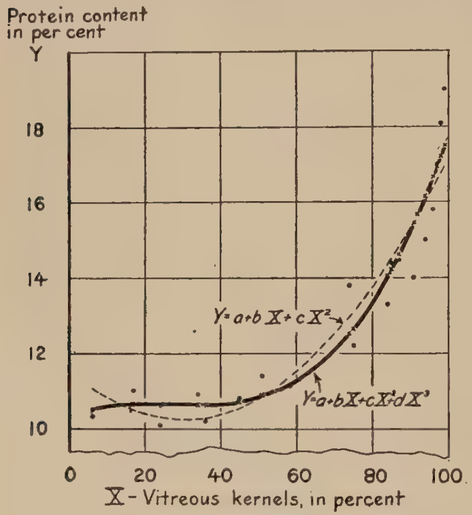

FIG. 14.—Dot-chart, with parabola and cubic parabola.

The last curve comes much closer to describing the relationship which actually exists than does the previous curve. Even so, however, it is not entirely satisfactory, for it gives estimates which are still too low at the very highest percentage of vitreous kernels. Except for this portion, and the downturn at the beginning, it seems quite satisfactory.

There are still other types of curves, however, some of which might give better fits than the ones we have tried. For instance the fourth-order parabola,

$$Y = a + bX + cX^2 + dX^3 + eX^4$$

can be fitted by an extension of the methods just described, as can parabolas with even more terms. Those are rarely useful, however, as the greater the number of terms, the greater the tendency becomes for the curve to "wiggle." In addition, the volume of arithmetic required becomes extremely burdensome—the computations for the fourth-order parabola involving powers of X up to X^8.

Some of the other types of curves mentioned on page 66, however, particularly types *b*, *c*, and *d*, involving logarithms; and type *e*, using reciprocals, may be fitted with relatively little computation. The methods of fitting one of each of these types may be shown for the present case, even though they may fail to give any better fit than the curves which have already been computed.

"Fitting" a Logarithmic Curve.—The three simple types of logarithmic curves, *b*, *c*, and *d*, may all be fitted by exactly the same method previously used in fitting a straight line, except that the logarithms of X, of Y, or of both together, are employed where otherwise the values of the variables themselves are used. Comparison of the straight-line formula with the logarithmic formula indicates how this is done.

If we use $\overline{Y}$ to represent the logarithms of the Y values, and $\overline{X}$ to represent the logarithms of the X values, then our equations will change as follows:

$$(b)\quad \log Y = a + bX, \text{ to } \overline{Y} = a + bX$$

$$(c)\quad \log Y = a + b \log X \text{ to } \overline{Y} = a + b\overline{X}$$

$$(d)\quad Y = a + b \log X, \text{ to } Y = a + b\overline{X}$$

In each case it is evident the new equation is identical in form with the simple straight-line equation,

$$Y = a + bX$$

and the same methods may therefore be used in determining the constants a and b as were used earlier in equations (8) to (11).

Some indication as to which one of the three logarithmic formulas will come nearest to fitting a given set of data can be obtained by converting both the X and Y values to logarithms, variables $\overline{X}$ and $\overline{Y}$, and then making dot charts of $\overline{Y}$ against X, of $\overline{Y}$ against $\overline{X}$, and of Y against $\overline{X}$. If one chart shows the dots falling in substantially a straight line, then the one of the three equations corresponding to that chart will give the most satisfactory fit.[6]

The first step in applying any one of the three logarithmic equations to the data of the wheat example is to work out the logarithms and construct the three dot-charts, to indicate which formula to use. The form of computation is shown in Table 20.

It should be noted that in working out the logarithms, nothing can be added or subtracted from any of the variables (except for rounding off decimals).[7] In all the previous work the protein had been stated as protein in excess of 10 per cent, but now the original per cent figures are used once more. That is because logarithms deal with

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[6] This is strictly true only if the "goodness of fit" is measured in terms of the logarithms used.

Logarithms may also be used with parabola of higher orders, such as:

$$\text{Log } Y = a + bX + cX^2$$

Such involved curves will not be considered at length in this book, however.

[7] After the logarithms are once computed, however, they can be "coded" by subtracting a constant or by division, just as other variables have been treated formerly, with the same effect on the final constants obtained.

TABLE 20

VARIABLES IN WHEAT PROBLEM, AND LOGARITHMS OF VALUES

Per cent protein Y	Per cent vitreous kernels X	Logarithms of Variables:*	
		Protein $\overline{Y}$	Vitreous kernels $\overline{X}$
10.3	6	1.013	0.778
12.2	75	1.086	1.875
14.5	87	1.161	1.940
11.1	55	1.045	1.740
10.9	34	1.037	1.531
18.1	98	1.258	1.991
14.0	91	1.146	1.959
10.8	45	1.033	1.653
11.4	51	1.057	1.708
11.0	17	1.041	1.230
10.2	36	1.009	1.556
17.0	97	1.230	1.987
13.8	74	1.140	1.869
10.1	24	1.004	1.380
14.4	85	1.158	1.929
15.8	96	1.199	1.982
15.6	92	1.193	1.964
15.0	94	1.176	1.973
13.3	84	1.124	1.924
19.0	99	1.279	1.996

* Logarithms to base 10.

relative values, and the relation of 1 to 2 is quite different from the relation of 11 to 12. All the previous equations have dealt with absolute values, or differences from the average; and the difference between 1 and 2 is of course just the same as the difference between 11 and 12.

Figure 15 gives the three dot-charts in which the three different ways of combining the logarithmic and actual values are shown. None of the three gives a ~~very close linear relation, but the one where~~ $\overline{Y}$ and X are plotted seems to come nearest. The equation

$$\log Y = a + bX, \text{ or } \overline{Y} = a + bX$$

will therefore be used.

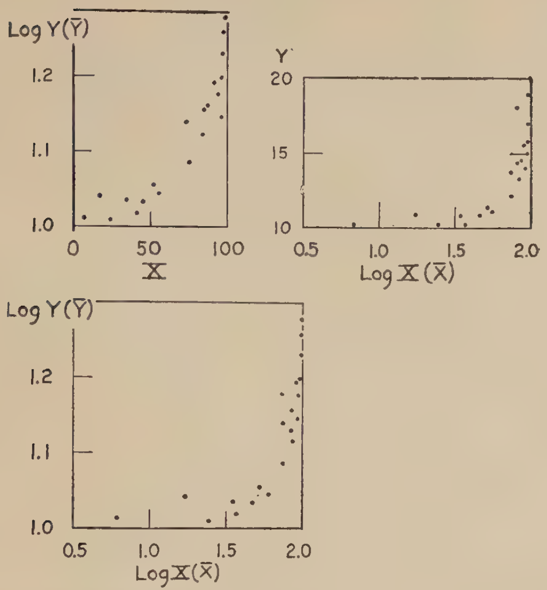

FIG. 15.—Dot-charts illustrating $\log Y = f(X)$; $Y = f(\log X)$; $\log Y = f(\log X)$.

The values necessary to determine a and b are as follows, using equations (9) and (10):

$$\Sigma X\overline{Y},\ M_x,\ M_{\bar{y}},\ \Sigma(X)^2$$

Table 21 shows in full the computation of these values from the original values of the two variables.

This computation gives the values necessary to compute a and b by formulas (9) and (10).

The averages of X and $\overline{Y}$ of course are:

$$M_x = \frac{\Sigma X}{n} = \frac{1{,}340}{20} = 67.0$$

$$M_{\bar{y}} = \frac{\Sigma\overline{Y}}{n} = \frac{22.389}{20} = 1.11945$$

Then

$$b = \frac{\Sigma(X\overline{Y}) - nM_xM_{\bar{y}}}{\Sigma(X^2) - n(M_x)^2} = \frac{1{,}545.888 - 20(67)(1.11945)}{107{,}566 - 20(67)^2} = 0.002576$$

and

$$a = M_{\bar{y}} - b(M_x) = 1.11945 - (0.002576)(67) = 0.9469$$

TABLE 21

Computation, for Wheat Problem, of Values Needed to Determine Constants for Logarithmic Curve

Per cent protein Y	Per cent vitreous kernels X	Logarithms of Y $\overline{Y}$	Extensions	
			X^2	$X\overline{Y}$
10.3	6	1.013	36	6.078
12.2	75	1.086	5625	81.450
14.5	87	1.161	7569	101.007
11.1	55	1.045	3025	57.475
10.9	34	1.037	1156	35.258
18.1	98	1.258	9604	123.284
14.0	91	1.146	8281	104.286
10.8	45	1.033	2025	46.485
11.4	51	1.057	2601	53.907
11.0	17	1.041	289	17.697
10.2	36	1.009	1296	36.324
17.0	97	1.230	9409	119.310
13.8	74	1.140	5476	84.360
10.1	24	1.004	576	24.096
14.4	85	1.158	7225	98.430
15.8	96	1.199	9216	115.104
15.6	92	1.193	8464	109.756
15.0	94	1.176	8836	110.544
13.3	84	1.124	7056	94.416
19.0	99	1.279	9801	126.621
Sums	$\Sigma X = 1340$	$\Sigma \overline{Y} = 22.389$	$\Sigma X^2 = 107{,}566$	$\Sigma X\overline{Y} = 1{,}545.888$

in terms of the variable, the equation required is therefore

$$\overline{Y} = a + bX = 0.9469 + 0.002576X$$

or

$$\log Y = a + bX = 0.9469 + 0.002576X.$$

The percentage of protein can now be estimated from the proportion of vitreous kernels observed for any sample of wheat, by substituting the per cent of vitreous kernels (the X values) in this equation and working it out. Thus for the first example, with 6 per cent of vitreous kernels, it would work out as follows:

$$\log Y = a + bX = 0.9469 + 0.0026(6)$$
$$\log Y = 0.9624.$$

Using a table of logarithms we find that the number corresponding to the logarithm 0.9624 (that is to say, its anti-logarithm) is 9.17. The estimated proportion of protein is therefore 9.17 per cent.

Similarly if the proportion of vitreous kernels in the second sample, 75, is substituted in the equation, the work to calculate the estimated proportion of protein is as follows:

$$\begin{aligned} \log Y &= a + bX = 0.9469 + 0.002576(75) \\ \log Y &= 1.1401 \\ \text{anti-log } 1.1401 &= 13.81. \end{aligned}$$

The estimated proportion of protein is therefore 13.81 per cent.

Table 22 shows this computation carried through for each of the 20 observations.

TABLE 22

Computation, for Wheat Problem, of Estimated Protein Content from Per Cent of Vitreous Kernels on the Basis of a Logarithmic Curve

($\log Y = 0.9469 + 0.00258\,X$)

Per cent vitreous kernels X	Estimated per cent protein		Actual per cent protein Y	Percentage errors in estimating protein proportion $100\left(\frac{Y}{Y'} - 1.00\right)$
	Estimated logarithm $\overline{Y}$	Anti-log of estimate Y'		
6	0.9624	9.2	10.3	+12.0
75	1.1401	13.8	12.2	−11.6
87	1.1710	14.8	14.5	− 2.0
55	1.0888	12.3	11.1	− 9.8
34	1.0345	10.8	10.9	+ 0.9
98	1.1993	15.8	18.1	+14.6
91	1.1813	15.2	14.0	− 7.9
45	1.0628	11.6	10.8	− 6.9
51	1.0783	12.0	11.4	− 5.0
17	0.9907	9.8	11.0	+12.2
36	1.0396	11.0	10.2	− 7.3
97	1.1968	15.7	17.0	+ 8.3
74	1.1375	13.7	13.8	+ 0.7
24	1.0087	10.2	10.1	− 1.0
85	1.1659	14.7	14.4	− 2.0
96	1.1942	15.6	15.8	+ 1.3
92	1.1839	15.3	15.6	+ 2.0
94	1.1890	15.5	15.0	− 3.2
84	1.1633	14.6	13.3	− 8.9
99	1.2019	15.9	19.0	+19.5

It should be noted in this table that errors made in estimating the proportion of protein are stated as relative errors rather than absolute errors. That is done because the thing that is really estimated is the logarithm of the percentages of protein, or $\overline{Y}$, and the errors are really the differences between the actual logarithms and the estimated logarithms. If z is used to stand for the error, in this case z is really in terms of logarithms, that is:

$$z = \log Y - \text{estimated } \log Y, \text{ or } \overline{Y} - \overline{Y}'.$$

or in terms of natural numbers:

$$\text{anti-log } z = \frac{\text{anti-log } \overline{Y}}{\text{anti-log } \overline{Y}'} = \frac{\text{actual } Y}{\text{estimated } Y}$$

Subtracting the constant 1.00 and multiplying by 100 changes this relative figure to the percentage which the observed value is above or below the estimate.[8]

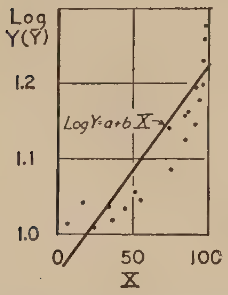

FIG. 16.—Dot-chart showing observations and fitted line for equation $\log Y = a + bX$, in logarithms of Y.

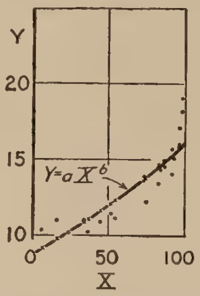

FIG. 17.—Dot-chart showing observations and fitted line for equation $Y = aX^b$, in natural values of Y.

In Figures 16 and 17 the actual proportions of protein, shown as dots, are compared with the estimated values as worked out by the logarithmic relation. In the first of these figures the actual and estimated values are both stated in terms of the logarithms. It is quite apparent here that this equation assumes a straight-line relation between the proportion of vitreous kernels and the logarithms of the proportion of protein; since they were computed by a straight-line equation ($\log Y = a + bX$) the estimated values all lie along the continuous straight line indicated. The next figure, however, compares the actual proportion of protein with the estimated, both stated in actual terms. Here the continuous curve which the logarithms produce in the estimated actual values is clearly shown. The relation between the proportion of vitreous kernels and the per cent of protein, as shown by this curve, does not agree with the actual relation as shown by the original observations even as closely as did the previous curves computed by means of parabolic equations.

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[8] The reason for making this distinction will be seen later on, when the question of measuring the accuracy of the estimate is taken up.

Before discussing other ways of expressing the curvilinear relation it might be well to discuss the procedure to determine the constants a and b if either of the other two forms of simple logarithmic equations were used.

If the equation $Y = a + b \log X$ is employed, then the form $Y = a + b\overline{X}$ is used.

The values which must be computed are

$$M_y,\ M_{\bar{x}},\ \Sigma(Y\overline{X}),\ \Sigma(\overline{X})^2$$

and the constants are determined from the equations

$$b = \frac{\Sigma Y\overline{X} - nM_yM_{\bar{x}}}{(\overline{X})^2 - n(M_{\bar{x}})^2}$$

$$a = M_y - bM_{\bar{x}}$$

Since the equation is in terms of Y itself, the estimated values, computed from the logarithms of X, will be directly in values of Y, and will not have to be converted to the anti-logarithms.

If the equation $\log Y = a + b \log X$ is to be fitted, then the form $\overline{Y} = a + b\overline{X}$ is used.

The values which will have to be computed are:

$$M_{\bar{y}},\ M_{\bar{x}},\ \Sigma\overline{Y}\,\overline{X},\ \Sigma(\overline{X})^2,$$

and the constants are determined from the equations

$$b = \frac{\Sigma\overline{Y}\,\overline{X} - nM_yM_{\bar{x}}}{\Sigma(\overline{X}^2) - n(M_{\bar{x}})^2}$$

$$a = M_{\bar{y}} - bM_{\bar{x}}$$

In this case the equation is in terms of $\overline{Y}$, the logarithms of Y, and the estimated values will therefore have to be converted from logarithms into natural numbers to show just what the relationship is, just as was done in the case that was worked out in detail earlier.

It is evident that no matter which one of the three logarithmic curves is employed, the arithmetic is exactly the same as in determining the simple straight line, with the exception of computing the logarithms and of substituting the appropriate logarithms where the actual values would otherwise be employed.

In cases where other modifications of the straight-line equation, such as type (*e*), are to be used, the process is to transform the equation to a linear form, then compute the constants just as before.

Thus the type

$$Y = \frac{1}{a + bX}$$

can be converted to the form

$$\frac{1}{Y} = a + bX,$$

or, letting $\frac{1}{Y} = Q$,

$$Q = a + bX$$

The computation can then be carried out in the usual way, and after the estimated values of Q, Q', are worked, converted back into Y values by the equation $Y' = \frac{1}{Q'}$.

Limitations of equations in describing relationships.—Up to this point an expression of the relation between the proportion of vitreous kernels and the proportion of protein in each sample has been worked out on the basis of a number of different mathematical formulas. Each different equation has given a different curve. Some, such as the cubic parabola or the logarithmic curve, have given curves coming somewhere near to the relationship shown by the actual observations themselves; others, such as the simple straight line, have entirely failed to describe the relation. Yet the exact slope or shape of each curve was determined from the same set of observations; the constants of each curve were determined by "fitting" the same data. The diversity in the shape of the different curves is strikingly shown in Figure 18, where the several different curves are all drawn on one scale, and the original observations are shown as well. It is quite apparent that the differences in the shapes of the several curves are due solely to the particular form of equation used in computing them. There are certain types of relations which can be accurately represented by each of these equations. When it is "fitted" to data where that type of relation is really present, then it can give a curve which accurately represents the true relation shown by the data. When, however, as in the present case, an attempt is made to represent a relation by an equation which does not truly express the nature of the relation, the resulting curve gives only a distorted representation of the true relation—*it shows the relation only insofar as it is possible to do so within the limits of the particular equation used.*

So far there has been no attempt to show what there is in the "nature" of relations which may make them of the type to be accurately represented by one type of equation or by another. Instead, the purely empirical test of the way each one fits has been relied upon. If, as judged by the eye, the relation shown by the fitted curve *looked like* the relation shown by the original observations, we have said it gave a satisfactory fit; if it has not looked like it, we have said it did not give a satisfactory fit. And in this particular case, none of the computed curves has been really fully satisfactory—we can readily see that there might be some other smooth continuous curve which would come much closer to the actual observations than does any of the curves so far computed.

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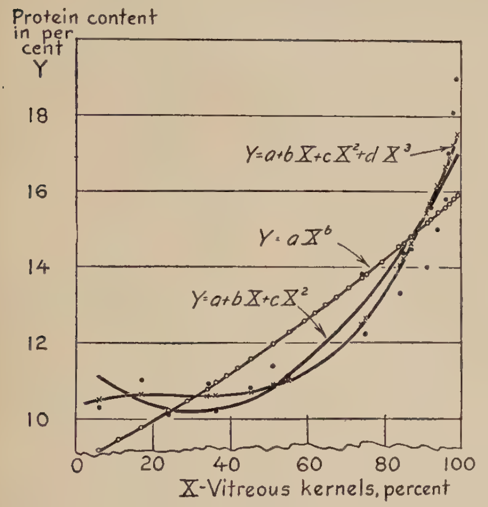

FIG. 18.—Original observations, and several different types of fitted curves.

Of course we might continue the process, using more and more complex equations, until finally we found one which did satisfactorily describe the relation. Or we might find that *no* ordinary mathematical expression would describe the relation. It might be that the underlying curve was so complex that it could not be represented in elementary algebraic terms. But even if we could describe the relation satisfactorily by some type of equation, the only advantage would be that then we would have some way of estimating values of the dependent variable (per cent of protein) from the independent variable (proportion of vitreous kernels) such as would agree reasonably well with the values actually observed. So long as the equation had been derived merely by the "cut and try" method described, it would have no meaning beyond serving as a simple device for estimating values of the one variable from known values of the other, and would throw no particular light upon the real or inherent nature of the relation. For if we could find, by enough trying, one equation which would represent the relation satisfactorily, it might be that we could also find another. As a matter of fact, sometimes it is found that two different types of equations may each give exactly the identical curve when figured out.[9] Which one expresses then the "true" nature of the relation? Merely because a given equation *can* reproduce a certain relation is no proof that it really "expresses" the nature of the relation. Something more must be known than merely that it *can* express the relation. What that something is will be taken up in a later section.

If, however, it is not desired to determine what the "real nature" of the relationship is, but it is merely desired to express it sufficiently well so that values of one variable (such as protein content) can be estimated from known values of another (such as the proportion of vitreous kernels), then it does not make any difference what type of equation is used, so long as it represents the observed relationship adequately. As a matter of fact, it is not really necessary to have an equation at all. If we have only a graph of the curve, or a table of values for one variable corresponding to values of another, from which we can construct a graph, that is all that is really necessary. For if we have a graph of the curve we can very readily estimate the value for one variable from corresponding known values for another by simply reading it off from the curve. Thus in Figure 13 the curve for the equation

$$Y = a + bX + cX^2$$

is shown. If we wish to estimate the per cent of protein for a sample having, say, 50 per cent of vitreous kernels, we need only to run up the line for $X = 50$, and note the value of Y corresponding to that point on the curve. In this case it is apparently about 10.8 per cent. Similarly, the estimates of the per cent of protein corresponding to any other per cent of vitreous kernels within the range covered by the curve may be read off directly from the curve. Further, by enlarging the chart and making the scale sufficiently detailed, the estimated values may be read off to any degree of accuracy that is desired—much more accurately, as a matter of fact, than our ability to determine the real relation usually justifies, as will be evident later on.

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[9] An example of this type may be seen in the bulletin, "What Makes the Price of Oats," by Hugh Killough, U. S. Department Agriculture Bulletin 1351. Here equations of the general type (*c*) and (*e*) were both found to yield almost identical curves.

In many cases—perhaps in the great majority of cases—simply the working expression of the relation may be all that is either needed or desirable. The "true relation" between the variables may be so involved that a very complex mathematical expression would be required to represent it properly. Even simple types of physical relations may require rather complex curves to represent them. In many cases, too, the knowledge of the causes of the relation may be so undeveloped that there is no real basis for expressing the relationship mathematically. The relation between vitreous kernels and per cent of protein would be an example of this type—very complex details of chemical content and physical and biological structure are probably responsible, so complex as to be quite beyond satisfactory reduction to mathematical expression. Yet the original observations undeniably indicate that there is some sort of definite relation. For many practical purposes it may be entirely satisfactory merely to know what the relationship *is,* without bothering at all with what it really means; and even in scientific study that may frequently be satisfactory as a first step, since in many cases it is essential to know what are the facts before trying to work out the reasons *why* they are as they are.

When the expression of the relation is not to be used except as an empirical basis for estimating values of the dependent variable from the independent, or for showing just what the relationship *is,* then the elaborate technique of determining the constants of a mathematical equation and working out the estimated values by the use of that equation becomes largely needless. For in many cases a curve can be determined with only a small fraction of the effort required in "fitting" a mathematical equation, yet which fits the data quite as well as any mathematical curve. In such cases the curve may afford quite as satisfactory a description of the relation and a basis for estimating one variable from the other as if elaborate computations had been made. This method is known as free-hand smoothing.

Expressing a Curvilinear Relation by a Free-hand Curve.—The process of determining a free-hand curve may be very simply illustrated. In fact, it has already been suggested in much of the previous discussion. The very simplest way to do it would be to plot the original observations on coordinate paper, just as has been shown so many times before, and then draw a continuous smooth curve through them by eye in such a way as to pass approximately through the center of the observations all along its course. Where the nature of the relation is indicated quite as closely by the original observations as it is in the wheat problem which we have been discussing, this might yield quite a satisfactory expression of the relation. In other cases, however, the observations might be more widely scattered, and the underlying relation might be more difficult to determine, so that different persons, drawing in the curves free-hand, might draw in rather different curves. Some method is therefore needed to give a greater degree of precision to the result, and to insure that the same data would yield the same result even in the hands of different investigators.

This stability of result can be secured by a relatively minor extension of the methods already discussed in the first illustration of a two-variable relationship—the automobile-stopping problem. There it was found that by classifying the observations in appropriate groups, the general nature of the relation could be expressed by an irregular line connecting the several group averages. All that is needed is some method of deriving a continuous smooth curve from

TABLE 23

Computation of Averages to Use in Fitting Free-hand Curve, for Wheat-protein Problem

	Vitreous kernels below 25 per cent		Vitreous kernels 25 to 49 per cent		Vitreous kernels 50 to 74 per cent		Vitreous kernels 75 to 100 per cent	
	Per cent vitreous kernels	Per cent protein	Per cent vitreous kernels	Per cent protein	Per cent vitreous kernels	Per cent protein	Per cent vitreous kernels	Per cent protein
	6	10.3	34	10.9	55	11.1	75	12.2
	17	11.0	45	10.8	51	11.4	87	14.5
	24	10.1	36	10.2	74	13.8	98	18.1
	....	....	....	....	....	....	91	14.0
	....	....	....	....	....	....	97	17.0
	....	....	....	....	....	....	85	14.4
	....	....	....	....	....	....	96	15.8
	....	....	....	....	....	....	92	15.6
	....	....	....	....	....	....	94	15.0
	....	....	....	....	....	....	84	13.3
	....	....	....	....	....	....	99	19.0
Totals....	47	31.4	115	31.9	180	36.3	998	168.9
No. cases.	3	....	3	....	3	....	11	....
Averages.	15.67	10.47	38.33	10.63	60.00	12.1	90.73	15.35

that irregular line. Smoothing out that irregular line, free-hand, is a very evident and simple method. At the same time, starting with the irregular line of group averages gives a certain stability to the process and insures that different persons would draw in the curve with about the same position and shape.

Applying the process to the wheat problem, the first step is to classify the data into appropriate groups according to the values of the independent variable, the proportion of vitreous kernels, and to determine the average per cent of vitreous kernels and of protein content for the observations falling into each group. The discussion of the automobile problem has shown that for the differences in averages to be significant, it is necessary for the groups to be large enough so that the averages would not vary erratically from group to group. In some cases a little experimenting might be necessary to determine what this size would be. In the present case, inspection of the dot-chart showing the original observations (Figure 12, page 72) indicates that a class interval of 25 per cent of vitreous kernels will give groups large enough to make the averages of protein content fairly stable from group to group.

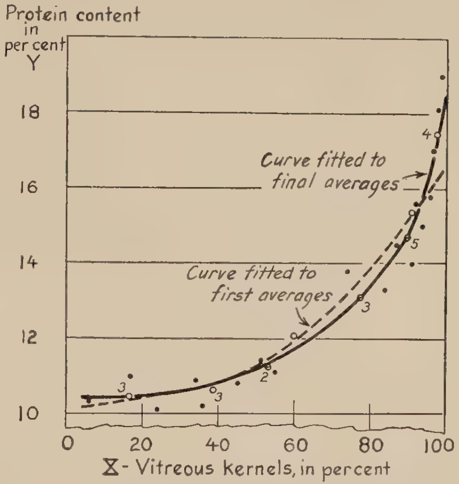

FIG. 19.—Original observations and averages of protein content, and free-hand curve.

The form of computation most convenient to obtain the group averages, using groups of the size suggested, is shown in Table 23.

The averages for the several groups are shown in Figure 19, indicated by hollow circles, while original observations are again shown by solid dots. A smooth continuous dashed curve has been drawn through the series of group averages, ignoring the individual observation and following only the general trend shown by the averages. This smooth curve comes quite near to representing the relation shown by the individual observations through most of its extent; but beyond 95 per cent of vitreous kernels it fails to follow the individual observations—through that portion of the range the protein content rises much faster than is indicated by the average for the whole range from 75 through 100 per cent vitreous kernels.

Over half of all the observations fall in this upper portion of the range, so it would seem reasonable to classify them into smaller groups so as to give them a better representation in indicating the slope of the curve. Let us try splitting the observation above 50 into four groups, each with about the same number of observations—say 50 to 69, 70 to 84, 85 to 94, and 95 to 100. The computation of the new averages is shown in Table 24.

TABLE 24

Computation of Sub-averages for Last Groups in Wheat Problem, for Fitting Free-hand Curve

	Vitreous kernels 50 to 69 per cent		Vitreous kernels 70 to 84 per cent		Vitreous kernels 85 to 94 per cent		Vitreous kernels 95 to 100 per cent	
	Per cent vitreous kernels	Per cent protein	Per cent vitreous kernels	Per cent protein	Per cent vitreous kernels	Per cent protein	Per cent vitreous kernels	Per cent protein
	55	11.1	75	12.2	87	14.5	98	18.1
	51	11.4	74	13.8	91	14.0	97	17.0
	....	....	84	13.3	85	14.4	96	15.8
	....	....	....	....	92	15.6	99	19.0
	....	....	....	....	94	15.0	....	....
Totals....	106	22.5	233	39.3	449	73.5	390	69.9
No. cases.	2	....	3	....	5	....	4	....
Averages.	53	11.25	77.67	13.1	89.8	14.7	97.5	17.48

These new averages, together with the previous ones for the lower groups, are also plotted in Figure 19, and the number of cases that each represents is indicated next to it, to aid in judging what weight to assign to that average. Finally, a smooth continuous curve has been drawn in, to pass as near as possible to the different averages without making illogical twists or turns. As is evident in the figure, it has been possible to draw the line with no point of inflection in it, yet it passes quite near to all the group averages and approximately through the middle of the individual observations. Further, the general course of the line is sufficiently well defined by the several group averages that if it were redrawn, either by the same person or another person, it could have only minor differences from the line actually shown.[10] Making the chart over two or three times, and drawing a separate curve on each trial, then averaging the two or three curves together, is one method of reducing the variation due to individual judgment in drawing the curve.

It is evident that the free-hand curve comes closer to agreeing with all the original observations than did any of the mathematically determined curves. So far as can be judged by eye alone, it "fits" the relation actually observed quite satisfactorily. So far as giving a definite statement of the relation, and serving as a basis for estimating values of one variable from known values of the other, this curve, obtained by the very simple process shown, is more satisfactory than any of the curves obtained by the mathematical computations.

The use of the free-hand curve in estimating values of the dependent variable, per cent of protein, from known values of the independent variable, proportion of vitreous kernels, may be readily illustrated. Taking the first observation, with 6 per cent of vitreous kernels, and reading off the corresponding proportion of protein from the curve in Figure 19, we get 10.4 per cent as the estimated protein content. Similarly for the second observation, 75 per cent vitreous kernels, the curve indicates 12.9 per cent as the proportion of protein. Reading off the estimated protein for each of the 20 observations we get the estimates shown in Table 25.

Even though in using the free-hand curve we do not have an

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[10] It should be noted that in drawing in the free-hand curve no attempt has been made to have the curve follow all the twists and turns of the irregular line of averages. As was shown previously with the automobile illustration, these irregular differences from group to group may very readily be due to chance fluctuations in sampling where the groups are small. Not unless the groups included a very much larger number of cases than these do here would one be justified in bending the curve because of the position of a single group-average, and not even then unless there was some reasonable basis for that shape of curve. In doubtful cases breaking up a particular group into smaller groups, as was just done in the wheat example, or reclassifying the observations into somewhat different groups, will help to determine whether or not an extra inflection should be put into the curve. It is also necessary to see if some single observation is responsible for the abnormality; if it is, it is better to disregard it and draw the curve without the extra twist. Only if the same shape of curve persists no matter which way the observations are grouped, and if it is not due to one or two exceptional observations, is one justified in drawing a curve of an apparently illogical shape—that is, of a shape that does not agree with what would reasonably be expected for the particular relationship involved.

TABLE 25

ACTUAL PER CENT OF PROTEIN AND PROPORTION ESTIMATED ON BASIS OF FREE-HAND CURVE

Proportion of vitreous kernels X	Actual proportion of protein Y	Proportion of protein estimated from vitreous kernels $Y' = f(X)$	Difference between actual and estimate $Y - Y'$
6	10.3	10.4	−0.1
75	12.2	12.9	−0.7
87	14.5	14.5	0
55	11.1	11.4	−0.3
34	10.9	10.7	0.2
98	18.1	17.4	0.7
91	14.0	15.2	−1.2
45	10.8	11.1	−0.3
51	11.4	10.3	1.1
17	11.0	10.5	0.5
36	10.2	10.8	−0.6
97	17.0	17.0	0
74	13.8	12.8	1.0
24	10.1	10.6	−0.5
85	14.4	14.2	0.2
96	15.8	16.7	−0.9
92	15.6	15.5	0.1
94	15.0	15.9	−0.9
84	13.3	14.0	−0.7
99	19.0	18.0	1.0

equation stating the relation between X and Y, we still have a mathematical expression of the relation between them. For we can write

$$Y' = f(X),$$

which simply means that the estimates, or Y' values, are *a function of* X; that is, for every X value there is some corresponding Y' value. Of course, we can find what this corresponding value is only by reading it off the curve; yet that is enough. We have a graphic statement of the functional relation; if we had a definite formula to represent the curve, then we would have an *analytical* statement of the relation as well.

But while the free-hand curve does not provide a definite equation expressing it, it is still possible to state the relation shown by the curve in other than graphic form. This can be done by constructing a table showing, for whatever values of the independent variable may be selected, the corresponding estimated values of the dependent variable. Such a tabular statement of the relation may be more readily comprehended by readers not accustomed to graphic presentation. Further, it provides a basis for reconstructing the curve on any scale desired for the purpose of making further estimates. Table 26 illustrates this method of stating the relation.

TABLE 26

Per Cent of Protein Corresponding to Various Proportions of Vitreous Kernels in Samples of Wheat, as Indicated by 20 Observations

Proportion of vitreous kernels	Corresponding proportion of protein	Proportion of vitreous kernels	Corresponding proportion of protein
Per cent	*Per cent*	*Per cent*	*Per cent*
10	10.4	70	12.4
20	10.5	80	13.5
30	10.7	90	15.0
40	10.9	95	16.2
50	11.2	99	18.0
60	11.7		

In the range where the curve is rising most steeply the readings are taken more closely together, to provide for more accurately reproducing that portion of the curve. In addition, no readings are taken beyond the range covered by the original observations, nor are any shown for the extreme ends where the observations are few. This raises the whole question of how curves like this can serve as a basis for estimating when measurements are made of the independent variable, such as proportion of vitreous kernels, in cases other than those used in determining the relation. This problem will be taken up at the end of this chapter. But first the question of whether to use free-hand or analytical curves will be discussed.

The Logical Significance of Mathematical Functions.—There has been frequent reference previously to the question whether an equation did or did not express "the real nature" of a relationship, without any explicit attempt to explain exactly what that meant. To know when we are justified in using the simple free-hand curve, and when we should go to the additional work of determining an equation for the curve, we must understand the logical bases for different types of equations, so that we can judge whether or not any particular type of curve can logically be expected to express the relation in any given set of observations.

The linear equation.—Many relations are so simple that we would not ordinarily think of expressing them mathematically. Thus, if a train is traveling 45 miles an hour, the distance traveled is equal to the time multiplied by the speed. Using t for the time in hours, d for distance, and s for speed, the relation is obviously

$$d = st$$

This is a simple straight-line relation. Now, if in addition, the train were a miles away from a given station at the beginning, after t hours of additional travel away from the station it would be D miles away, where

$$D = a + d = a + st$$

This is now expressed in the usual form for the straight-line equation, $Y = a + bX$. This equation is therefore the one to be used when it can logically be expected that each unit change in X causes a corresponding change in Y, regardless of the size of X. Thus in computing the distance the train has traveled we are assuming that it continued to travel at a definite rate, say 45 miles an hour, the whole way, and traveled the 200th mile just as fast as the first mile. Now if we were dealing with something where the change in Y was not the same for different values of X, the equation would no longer be satisfactory. For example, an airplane on a long-distance flight has to carry a heavy load of gasoline at the start and hence cannot attain full speed; the farther it goes the lighter its load becomes, and the higher speed it can make. In such a case the straight-line formula would not be applicable, since the speed of the plane would increase with the distance it had gone. If the straight-line formula were used, it would indicate that it would take just as long to travel the first hundred miles as the last hundred, whereas actually it would take longer than that to travel the first hundred and less than that to travel the final hundred. Only an equation which included some value that properly took into account the change in speed with the change in distance could satisfactorily represent this relation.

The quadratic equation.—Another case in which the rate at which Y increases changes as the value of X increases is that of a weight falling to the ground. Since the attraction of the earth is for practical purposes a constant, it exercises a constant pull on a falling body. Thus, the farther a body falls, the faster it travels. It is just as if in throwing a ball, a boy did not let go the ball for it to travel by its momentum, but was able to keep shoving against it, adding more and more speed to the momentum it already had. Physicists express this relation by saying that the velocity with which an object falls is accelerated at a constant rate. This equation, therefore, is:

$$V = gt$$

where g is a constant measuring the force of gravity, V is velocity in feet per second, and t is time in seconds.

With regard to the distance a body will fall in any given time, therefore, the case is much the same as with our airplane. The velocity, or speed, is increasing with every passing moment, and therefore the distance traveled in each succeeding second will be greater than the distance traveled in the previous second.

If we assume that the value of g in the equation is already known to be 32, the equation

$$V = gt$$

can then be written

$$V = 32t$$

We can then estimate the distance traversed by a falling body in each successive second by a process of approximation like this:

Let us figure that the average speed for each 2 seconds is the same as at the midpoint (which may not be exactly right) and then let us estimate the distance traversed in those 2 seconds by multiplying this average speed by the time. Then by adding all the distances together we can get an approximation of the total distance.

First we need to calculate the average speed for each period, using the last equation, $V = 32t$:

End of 1st second, speed $= 32(1) = 32$ = average speed for 1st two seconds
End of 3d second, speed $= 32(3) = 96$ = average speed for 2d two seconds
End of 5th second, speed $= 32(5) = 160$ = average speed for 3d two seconds
End of 7th second, speed $= 32(7) = 224$ = average speed for 4th two seconds
End of 9th second, speed $= 32(9) = 288$ = average speed for 5th two seconds

Then we can estimate the distance traveled in each 2-second period, as follows:

Period	Average speed, feet per second	Distance in that period, feet
1st	32	64
2d	96	192
3d	160	320
4th	224	448
5th	288	576
Estimated total distance..............		1600

$D = 16t^2$
$= 16 \cdot 100$
$= 1600$

Another estimate could be obtained by estimating the distance for each second separately, for there might be less error in assuming the speed at the middle of each second would represent the average for that second. On this basis the problem would work out.

Speed at middle of 1st second $= 32(\frac{1}{2}) = 16$; distance in that second $= 16$
Speed at middle of 2d second $= 32(1\frac{1}{2}) = 48$; distance in that second $= 48$
Speed at middle of 3d second $= 32(2\frac{1}{2}) = 80$; distance in that second $= 80$
Speed at middle of 4th second $= 32(3\frac{1}{2}) = 112$; distance in that second $= 112$
Speed at middle of 5th second $= 32(4\frac{1}{2}) = 144$; distance in that second $= 144$
Speed at middle of 6th second $= 32(5\frac{1}{2}) = 176$; distance in that second $= 176$
Speed at middle of 7th second $= 32(6\frac{1}{2}) = 208$; distance in that second $= 208$
Speed at middle of 8th second $= 32(7\frac{1}{2}) = 240$; distance in that second $= 240$
Speed at middle of 9th second $= 32(8\frac{1}{2}) = 272$; distance in that second $= 272$
Speed at middle of 10th second $= 32(9\frac{1}{2}) = 304$; distance in that second $= 304$

In 10 seconds, total distance traversed $= 1600$

This comes out exactly the same as before. On reflection, it is evident that this is to be expected. Since the velocity increases at *a uniform rate for each moment of time,* the true average rate of speed for any period will be just half-way between the speed at the beginning and at the end.[11] If we consider our 10 seconds as a whole, the velocity at the beginning is equal to

$$V = 32(t) = 32(0) = 0,$$

that is, the initial velocity is zero; whereas the velocity at the end is

$$V = 32(t) = 32(10) = 320$$

The average speed for the period, therefore, is

$$\frac{0 + 320}{2} = 160,$$

which is exactly the same as the speed at which the body is falling at the middle of the period, at the end of the 5th second, which is

$$V = 32(t) = 32(5) = 160$$

Computing the total distance traversed by multiplying the total time by this average speed, we have

$$d = (160)(10) = 1600,$$

giving exactly the same answer as our earlier computation.

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[11] This would not be true of all types of relations. If, for example, velocity increased at a *changing* rate, the smaller the units taken, the more accurate would be the result.

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The average speed during any period of t seconds is therefore $32t/2$. The total distance traversed in the t seconds can therefore be determined by multiplying the average speed, $32t/2$, by the total number of seconds, t. This gives

$$d = 32\left(\frac{t}{2}\right)(t)$$

or

$$d = 32\frac{t^2}{2}$$

$$= 16t^2$$

So far, we have assumed that we know the acceleration, or rate of increase in velocity per second. Suppose instead we had not known it to begin with: How could we have found it out?

If we used the symbol g to represent this value, we could have carried out all the previous calculations, except that we would have used "g" where instead we have used "32."

Our last formula then would have been

$$d = g\frac{t^2}{2}$$

or

$$d = \frac{g}{2}t^2$$

Letting $\frac{g}{2} = b$, the equation then would read

$$d = bt^2$$

We could very readily determine the value for b by observing the distance a given body falls in 1 second, in 2 seconds, in 3 seconds, etc., and then working out the probable value for the constant, just as has been done before.

After we had made measurements of several distances d in the several periods t, we could determine b most readily for the straight-line equation by using T for t^2. Then

$$d = bT$$

(which is the same form as $Y = a + bX$).

Since we may assume $a = 0$, it follows from equation (10),

$$a = M_y - bM_x,$$

that

$$0 = M_y - bM_x;$$

hence,

$$bM_x = M_y$$

and

$$b = \frac{M_y}{M_x} = \frac{\Sigma(Y)}{\Sigma(X)}$$

or, in the terms of this particular example,

$$b = \frac{M_d}{M_T}$$

which gives a basis for determining g, the acceleration due to gravity in feet per second, simply by making observations of the time for bodies to fall varying distances.

Substituting an observation of 64 feet in 2 seconds in this equation, it gives $b = \frac{64}{2} = 32$; hence $g = 32$.

In this case it should be noted that the formula

$$d = \frac{g}{2}t^2$$

is derived on the assumption that the attraction of gravity is a constant, tending to increase velocity at a uniform rate per second, or other unit of time. Only if this assumption is correct can the equation be used. The equation is directly based upon this assumption; the reasoning used in deriving the equation also serves to explain what the constants obtained *really represent.* On the basis of this reasoning the equation determined is not a mere empirical expression of the relation between time falling and distance traversed. Instead, it is a fundamental measurement of *why that distance is what it is,* and relates it in a logical manner to the attraction of the earth.

While it would be quite possible in this particular case to draw a free-hand curve expressing the relation between time and distance, it would not be so satisfactory as the mathematical equation. The curve would merely state what the relation was; the equation, in addition, explains *why* it is.

The parabolic equation.—Another physical case in which a definite relationship may be established logically, and then measured statistically, is in the case of a projectile fired from a gun.

Disregarding the resistance of the air, there are three elements which will determine the height the projectile will have reached at any given instant after it leaves the muzzle of the gun. The simplest of these elements is the height of the muzzle of the gun itself, represented by a in Figure 20. All the latter changes in elevation will obviously have to be added to that.

The second element is the rate at which the projectile is moving upward at the instant it leaves the muzzle. That is dependent, of course, on the angle at which the gun is elevated, and the muzzle velocity. If the gun were elevated 1 per cent from the horizontal, and the muzzle velocity were 1,000 feet per second, the projectile would leave the muzzle moving upward at the rate of 10 feet per second. If there were no resistance of the air, and if there were no force of gravity to pull the projectile off its course, its momentum would carry it on in this direction to infinity, as illustrated by the straight line in the picture. Here b represents the increase in elevation the projectile would attain for each additional second of flight, and a and bt the elevation it would attain if gravity did not influence it.

But gravity is at work too. As we have already seen, as soon as a body is released, the pull of gravity tends to move it downward at ever increasing speed. Even if it is headed upward as when shot from a gun, the pull of gravity starts tending to pull it down. The diagram illustrates what happens, using C to represent the distance the body would have fallen if it had no upward velocity. At first the gain in height from its upward momentum is more than enough to offset the tendency to lose height because of the pull of gravity, and the projectile moves upward along the curved course indicated. But finally the loss due to gravity becomes greater than the gain from its original upward momentum and the trajectory gradually turns downward, until the projectile finally comes to rest in the earth or on its target.

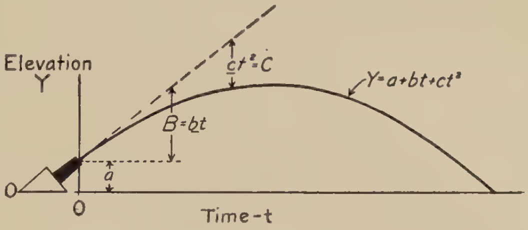

FIG. 20.—The trajectory of a projectile, illustrating the equation $Y = a + b\,X + c\,X^2$.

The height that the projectile reaches at any moment is the sum of these three components—the original height, the upward course, and the loss by gravity. Its height, then, can be expressed by adding together the three elements.

a remains the same, regardless of the time elapsed.

B, the height due solely to the original momentum, depends on the time, increasing as the time increases. If we let b represent the initial rate of gain in elevation per second of time, B can then be stated:

$$B = bt$$

Finally, C depends on the time elapsed, and, as we have just seen, varies with the square of time. Using the same notation as in our falling stone problem, but substituting C for distance fallen

$$C = -\frac{g}{2}t^2 = ct^2$$

Adding these three elements together, we obtain the equation for the height of the projectile at any instant, letting H represent height in feet.

$$H = a + bt + ct^2$$

It will be seen that this equation is exactly identical in form with the equation for a parabola

$$Y = a + bX + cX^2$$

If measurements had been made for a given gun, firing the same charge at the same elevation of the gun, of the height of the projectile at various given times after firing the charge, that would give a series of X and Y values which could be used in computing the constants a, b, and c, even if all were unknown to start with.

If the equation were actually worked out, it would tell much more than would merely the graph of the relation. For if the reasoning on which the several different constants were included in the equation was correct, then the equation would furnish a real explanation of why the projectile moved as it did, in terms of the laws of motion and of gravity upon which all such movements depend.

Reasoning such as this, carried out to much greater lengths, has formed the basis for the scientific "laws" which have been discovered in physics and chemistry, and expressed in definite equations. The methods for determining the constants in such equations, as presented earlier in the chapter, were first devised to serve in determining such types of relations. But when the same methods are applied to biological, economic, educational, or other relationships in the natural or social sciences, their value is much more limited. Only rarely is there real basis for expecting a particular mathematical relationship such as can be expressed in a given type of equation. In many cases our knowledge of the reasons for the relationship are altogether too limited to enable us to say *why* the relationship is; and even where we can establish the reasons, they are frequently too complicated or too involved—or even too biological—to admit of mathematical treatment. If we express a given relation by a formula, merely on the basis that that formula seems to describe the observed relation satisfactorily, we do not have any greater knowledge of the relation than if we merely drew in a free-hand curve. The equation is simply an empirical description of the relation; of and by itself, it offers no clue as to what the relation means.

When to fit a mathematical equation.—From this discussion, the following tentative conclusion may be reached: Only when there is some good logical basis for expecting a certain type of relation to hold should mathematical curves be employed in describing the relationship. When there is a logical basis for using a given formula, the constants of the equation serve as an explanation of the real nature of the relationship. In all other cases the mathematical curve has no more significance than the free-hand curve; the latter may therefore be employed to describe the nature of the relation, and can be determined with much less expenditure of effort. That does not mean that a mathematical curve, based on adequate logical analysis, is of no additional value. If it can be shown that such a curve does fit the data, that may verify an hypothesis and so provide a "law" to state the nature of the relationship, which may be of far more value than the mere empirical statement of what the relationship is observed to be. If, however, there is no logical basis for anything except the empirical statement of the observed relation, then the free-hand curve is just as valuable as is one fitted by aid of a mathematical equation.

A mathematical equation used in an economic problem.—Economists have frequently advanced the idea that for any one commodity there will tend to be a constant relation between the rate of change in the quantity consumers would buy and the rate of change in price. That is, if an increase of, say 1 per cent in price would cause a 2 per cent decrease in consumption when prices were low, a similar increase of 1 per cent in price would still cause a decrease of 2 per cent in consumption even when prices were high and consumption was already low.

This economic hypothesis can be stated in definite mathematical terms quite as readily as the various physical hypotheses which have been mentioned; for it makes certain definite assumptions as to the precise way the two variables (price and consumption) are related.

Using C for quantity consumed and P for price, the statement says that the relation

$$C = f(P);$$

that is, that the quantity consumed depends upon and varies with price, is a function of the type

$$C = kP^b$$

The reason for its being that type can be seen by stating the last equation in logarithmic form:

$$\log C = a + b \log P$$

This says now that a given change in the logarithm of P is always accompanied by a change of b times as much in the logarithm of C. Remembering that the same absolute change in the *logarithm* of a number always means a constant *percentage* change in its actual value, we can see that this equation states the economic hypothesis that a given proportional change in price is always accompanied, on the average, by a constant proportional change in consumption, no matter whether price was high or low to start with.

The practical application of the logarithmic demand equation may be illustrated by a concrete case. Table 27 shows the actual slaughter of hogs in the United States during the years 1922 to 1927, and the average price paid by packers during those years. If we assume that all the hogs produced were consumed, and ignore any possible shifts in the levels of demand during that period, we may ask whether the relation between the annual average price and the consumption of

TABLE 27

Slaughter of Hogs, and Average Price, and Computation of Logarithmic Curve

($\log C = a + b \log P$)

Year*	Weight of hogs slaughtered (C)	Price of hogs† (P)	Logarithms of data		Extensions	
			Slaughter	Price	$\overline{CP}$	$\overline{P}^2$
	Billion pounds	*Dollars per cwt.*	$\overline{C}$	$\overline{P}$		
1922–23	11.66	7.62	1.0667	0.8820	0.94083	0.77792
1923–24	11.83	7.61	1.0730	0.8814	0.94574	0.77687
1924–25	10.25	10.71	1.0107	1.0298	1.04082	1.06049
1925–26	9.66	12.16	0.9850	1.0849	1.06863	1.17701
1926–27	10.04	10.84	1.0017	1.0350	1.03676	1.07123
1927–28	10.99	9.20	1.0410	0.9638	1.00332	0.92891
Sums.......	....	....	6.1781	5.8769	6.03610	5.79243

* From November to October, inclusive.

† Adjusted for differences in price level, to 1928 level.

hog products in the United States during this period agrees with the hypothesis that a given proportional fall in price causes a constant proportional rise in consumption.

Accordingly we "fit" the equation

$$\log C = a + b \log P,$$

(where C = consumption, and P = price)

to the data by the methods previously discussed. The actual computations are all shown in Table 27.

$$M_{\bar{c}} = \frac{\Sigma \bar{C}}{n} = \frac{6.1781}{6} = 1.02968$$

$$M_{\bar{p}} = \frac{\Sigma \bar{P}}{n} = \frac{5.8769}{6} = 0.97948$$

$$\Sigma(\overline{cp}) = \Sigma(\overline{CP}) - nM_{\bar{c}}M_{\bar{p}}$$

$$= 6.03610 - 6(1.02968)(0.97948) = -0.01521$$

$$\Sigma(\bar{p}^2) = \Sigma(\bar{P}^2) - nM_{\bar{p}}^2 = 5.79243 - 6(0.97948)^2 = 0.03614$$

$$b_{\overline{cp}} = \frac{\Sigma(\overline{cp})}{\Sigma(\bar{p}^2)} = \frac{-0.01521}{0.03614} = -0.42086$$

$$a_{\overline{cp}} = M_{\bar{c}} - bM_{\bar{p}} = 1.02968 - (-0.42086)(0.97948)$$

$$\bar{C} = a_{\overline{cp}} + b_{\overline{cp}}\bar{P}$$

$$= 1.4419 - 0.42086\bar{P}$$

$$\log C = 1.4419 - 0.4209 \log P$$

We may next test how well this equation describes the relationship, by plotting both the original observations and the curve corresponding to the equation. Figure 21 shows this comparison in terms of the logarithmic values used in the computation, and with the logarithmic values of the function (which, of course, is a straight line). It is seen that this straight line seems to fit the original values quite closely; they fall very close to it, above and below, in such a random fashion that no other type of curve seems necessary.

The comparison may also be made in terms of the original values, using the estimated values of the curve transformed from logarithms back to real numbers. Figure 21 shows the comparison of these values. Here again, the demand curve is seen to be a satisfactory "fit" to the actual data.

The economic hypothesis as to the relation between price and consumption would therefore seem to be borne out so far as this particular illustration is concerned, and with the assumptions stated. The size of the constant, b, -0.42, indicates that anywhere along the curve a one per cent increase in the price of hogs is accompanied by approximately 0.4 per cent change in hog consumption in the opposite direction, or *vice versa.*

The wheat-protein example, on the other hand, illustrated a case where there was no logical basis for the use of any particular equation, and where a free-hand curve was therefore as satisfactory as any other type and gave a better fit than any of the analytical types which were tried. As has been stated, the great majority of the problems in the natural and social sciences are probably of this type, where the relation can be measured even though the specific causes for it cannot be stated in mathematical language. Only where the relations can be explained on some logical basis which lends itself to mathematical statement is there justification for a large amount of work to "fit" a specific formula; and even then, if it is found that that particular formula does not give as good a "fit" as a simple free-hand curve, there would be question as to whether the hypothesis were in agreement with the facts in that particular case.

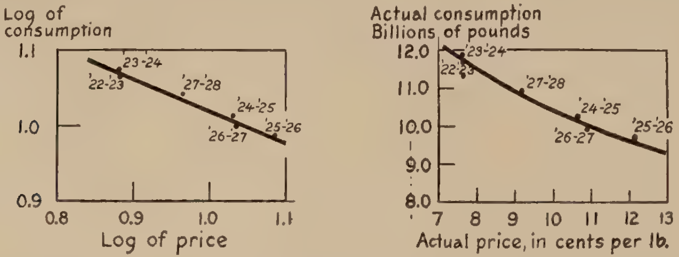

FIG. 21.—The relation of consumption of hog products to hog prices, fitted by a logarithmic demand curve; both in logarithms of consumption and in natural numbers.

Limitations in Estimating One Variable from Known Values of Another.—The methods shown so far provide a definite technique by which an investigator can determine the way in which the values of one variable differ as the values of another related variable differ. These same operations afford a basis for estimating values of the dependent variable from given values of the independent variable, for cases in addition to those from which the functional relation was determined. Whether such estimated values, for cases not included in the original study, can be expected to agree with the true values if they could be determined, depends upon two groups of considerations: (a) the descriptive significance of the curve, and (b) its representative significance when it comes to applying to new observations.

These two groups of considerations apply (a) to exactly what a given curve means, with regard solely to the particular cases from which it was determined; and (b) the significance of the curve with regard both to the ability of those observations to represent the universe (whole group of facts) from which they were drawn, and the ability of the curve to represent the true relations existing in that universe. This second group involves an extension of the points which were raised in the first chapter as to the reliability of an average; discussion of these questions will be deferred to Chapter 18.

The representative significance of a curve depends upon the number of observations from which its shape was determined, and how closely the curve as determined "fits" those observations. Since the number of observations usually differs along the different portions of a curve, it may be much more reliable in its central portions, where the bulk of observations occurs, than in the extreme portions where the number of observations may be much less. This may be especially marked in the case of complex curves fitted by mathematical means, where single extreme observations may have a material effect upon the shape of the end portions. In any event, only those portions of the curve where there are enough observations to make its shape and position definite should be regarded as statistically determined; the end portions, when dependent upon a few observations, should either not be used at all, or else stated as very rough indications of the true curve.

It is particularly to be noted that determination of the line or curve of relationship gives no basis for estimating beyond the limits of the values of the independent variable actually observed. No matter whether a formula has been fitted or not, any attempt to make estimates beyond the range of the original data by "extrapolation," i.e., by extending the curve beyond the range of the observed data, gives a result that is not based on the statistical evidence. In case a formula has been used which has a good logical basis, extrapolation may give a result which it is logical to expect—but its reasonableness rests on the validity of the logic, rather than on a statistical basis.

The statistical analysis indicates only what the relations are within the range of the observations which are used in the analysis.

The "closeness" with which the line or curve fits the original data is another criterion of the reliance which can be placed in it. If the data all fall quite close to the line, that fact inspires more confidence in it than if they differ widely and erratically from it. But there are special statistical measures of just what this "closeness" is, and they will be given separate considerations in the next chapter.

It should be noted here, however, that many more cases are required to determine a relation with any degree of dependability than were used in the hog-consumption example just considered. That example was given to illustrate the type of problem where a definite equation might be applied, but not as an illustration of a real research problem.

Summary.—In some functional relations, the change in the dependent variable with changes in the independent variable can not be represented by a straight line. Such a relation may be represented by a curve showing the value of the dependent variable for each particular value of the independent variable. Curves may be fitted to given sets of observations either by use of mathematical functions, such as parabolas, logarithmic curves, hyperbolas, etc., or by various processes of free-hand smoothing. When there is some logical basis for the selection of a particular equation, the equation and the corresponding curve may provide a definite logical measurement of the nature of the relationship. When no such logical basis can be developed, a curve fitted by a definite equation yields only an empirical statement of the relationship, and may fail to show the true relation. In such cases a curve fitted free-hand by graphic methods may be even more valuable as a description of the facts of the relationship, than a definite equation and corresponding curve selected empirically.

In any event, estimates of the probable value of the dependent variable can not be made with any degree of accuracy for values of the independent variable beyond the limits of the cases observed; and can be made most accurately only within the range where a considerable number of observations is available. It may be possible to extrapolate the curve if its equation is based on a logical analysis of the relation as well as on the cases observed; but in that case the logical analysis, and not the statistical examination, must bear the responsibility for the validity of the procedure.

CHAPTER 7

MEASURING ACCURACY OF ESTIMATE AND DEGREE OF CORRELATION

The methods developed up to this point may be used to estimate the values of one variable when the values of another are known or given. They also furnish an explicit statement of the average difference or change in the values of the estimated or dependent variable for each particular difference or change in the value of the known or independent variable. But that is not enough. In addition it is frequently desirable to answer two queries: (1) How closely can values of the dependent variable be estimated from the values of the independent variable? and (2) How *important* is the relation of the dependent variable to the independent variable? Special statistical devices, termed (1) the *standard error of estimate* and (2) the *coefficient* and *index of correlation*, have been developed to meet these two needs.

The Closeness of Estimate—Standard Error of Estimate.—Attention has previously been called to the fact that when some dependent variable, such as the distance required for an automobile to stop after the brake is applied, or the protein content in wheat samples, is estimated from another variable, such as the speed at which the car is moving, or the proportion of vitreous kernels in the sample, the estimated values in many cases will not be the same as the values of the dependent variable which were originally observed. These differences are obviously due to *residual* causes; that is, to variations in the dependent variable which were not "dependent" on the particular independent variable which had been determined. For that reason the differences between the estimated values and the actual values are termed residual differences, or more simply, *residuals.*

For linear relations.—The meaning of the residuals and their use in determining the standard error of estimate and the coefficient and index of correlation can best be understood if illustrated by a concrete case. Such an illustration is given in Table 28. Here 18 observations of the number of days (X) that horses worked on different farms, and the quantity of grain fed each horse (Y), have been fitted by a straight line to estimate the quantity of feed from the days of work. The estimated quantities, Y', and the residuals, z, or differences between the estimate and the actual, are also shown.

TABLE 28

DAYS WORKED BY HORSES, GRAIN FED PER HORSE, AND GRAIN ESTIMATED FROM DAYS OF WORK

Days worked X	Grain fed, in hundred weight Y	Estimated grain fed* Y'	Excess of actual over estimate z
107	49	48.0	1.0
70	28	40.9	−12.9
81	44	43.0	1.0
57	36	38.4	− 2.4
87	58	44.2	13.8
114	38	49.4	−11.4
73	49	41.5	7.5
74	53	41.7	11.3
42	33	35.5	− 2.5
90	45	44.8	0.2
100	59	46.7	12.3
59	39	38.8	0.2
86	38	44.0	− 6.0
89	41	44.6	− 3.6
98	42	46.3	− 4.3
95	45	45.7	− 0.7
76	39	42.1	− 3.1
98	46	46.3	− 0.3

* Computed by regression formula $Y = 27.43 + 0.1927\,X$.

The residuals vary from +13.8 to −12.9. If we wish to say how large they are on the average, we can ignore the plus and minus signs and compute the average deviation. For the 18 residuals in Table 28, the average deviation is 5.25, and the standard deviation is 7.13. If these residuals are grouped in a frequency distribution, they fall as shown in Table 29.

The standard deviation of z is different from the standard deviations previously computed. Instead of showing the standard deviation of grain fed from the mean quantity (that is, σ_y), it shows the standard

TABLE 29

FREQUENCY DISTRIBUTION OF RESIDUALS IN ESTIMATING GRAIN FED

Residual*	Number of times occurring	Residual*	Number of times occurring
−16 to −12	1	0 to + 4	4
−12 to − 8	1	+ 4 to + 8	1
− 8 to − 4	2	+ 8 to +12	1
− 4 to 0	6	+12 to +16	2

* As stated in Chapter 1, −12 to −16 means from −16 up to, but not including, −12; and so on for the other groups.

deviation around a changing quantity, depending on the number of days worked. The σ_z is thus the standard deviation around the fitted line of relation, and may be indicated graphically on a correlation chart as a certain area above and below the fitted line.

The standard deviation is 7.13, so we would expect two-thirds of the residuals to come between +7.13 and −7.13. Of the 18 cases, 12 came within this range of the line, or 67 per cent of all the cases. Similarly, only 5 per cent of the cases would be expected to fall outside the range $+2\sigma$, or below −14.3 or above +14.3. Actually, none come outside this range, which is close to the expected proportion, with this limited number of observations.

Where the same set of conditions prevails as those under which the original data were selected and only the independent variable is known, it may be desired to estimate the probable value of the dependent variable from the known value of the independent. Thus if the number of days that horses work on other farms in the same area is known, it may be desired to estimate the quantity of grain that will be needed to feed them. Or in a case where yield of cotton with various applications of irrigation water has been determined (note the example in the next chapter) it may be desired to estimate the most probable yield on other fields, solely from the amount of water applied. In case the estimates were to be made for new samples taken from the same "universe"—for example, on the same soil type, in the same area, and for the same year—as were the previous samples, a knowledge of the standard deviation of the residuals for original samples gives a basis for judging how closely the new estimates are apt to approximate the true, but unknown, yields for the new samples. Similarly in the feeding case it is evident that the errors of estimate will not often be greater than 14.3 cwt. of grain, and usually will be less than 7.1 cwt.

Since the standard deviation of the residuals does thus serve to indicate the closeness with which new estimated values may be expected to approximate the true but unknown values, it has been named the *standard error of estimate,* since it has the same meaning in indicating the reliability of an estimated value based on previously observed relations that the *standard error of an average* has in indicating the confidence to be placed in the reliability of a particular average.

The symbol S is used to denote the standard error of estimate; $S_{y \cdot x}$ indicating the standard error for estimates of Y made from a linear relation to X, by the equation $Y = a + bX$. Similarly, $S_{y \cdot f(x)}$ would indicate the standard error for estimates of Y made on the basis of a free-hand curve relation to X, as indicated by the equation $Y = f(X)$.

The standard error of estimate is therefore defined by the two equations:

$$\left.\begin{aligned} S^2_{y \cdot x} &= \frac{\Sigma z^2}{n-1} \\ S^2_{y \cdot f(x)} &= \frac{\Sigma (z'')^2}{n-1} \end{aligned}\right\} \qquad (20)$$

$N - j + 1$

$18 - 2 + 1 = 17$

(Here $n - 1$ is used in calculating the standard error of estimate for the same reason that it was used, in Chapter 2, in calculating the standard error of an average.)

The standard error of estimate in estimating grain fed the horses from number of days worked, by the linear equation is therefore

$$S^2_{y \cdot x} = \frac{18(7.13)^2}{17} = 53.83$$

$$S_{y \cdot x} = 7.34$$

For curvilinear relations.—The calculation of the standard error where a curvilinear function is used to express the relation may also be illustrated by the horse-feeding data. Fitting a free-hand curve by methods already described, estimates of Y from the relation $Y = f(X)$ were obtained, as shown in Table 30.

The standard deviation of the new residuals is 6.85. Comparing this to the standard deviation of 7.13 for the residuals from the straight line, it appears that the curve, on the average, lies closer to the original observations than does the straight line.

TABLE 30

DAYS WORKED BY HORSES, GRAIN FED PER HORSE, AND GRAIN ESTIMATED FROM DAYS OF WORK, BY FREE-HAND CURVE

Days worked X	Grain fed, in hundredweight Y	Estimated grain fed Y''	Excess of actual over estimate z''
107	49	46.5	2.5
70	28	41.4	−13.4
81	44	44.2	− 0.2
57	36	37.4	− 1.4
87	58	45.5	12.5
114	38	46.5	− 8.5
73	49	42.2	6.8
74	53	42.5	10.5
42	33	32.5	0.5
90	45	45.9	− 0.9
100	59	46.5	12.5
59	39	38.1	0.9
86	38	45.2	− 7.2
89	41	45.8	− 4.8
98	42	46.5	− 4.5
95	45	46.4	− 1.4
76	39	43.0	− 4.0
98	46	46.5	− 0.5

Working out the standard error of estimate for the curvilinear function by equation (20)

$$S^2_{y \cdot f(x)} = \frac{\Sigma(z'')^2}{n-1}$$

$$= \frac{18(6.85)^2}{17} = 49.68$$

$$S_{y \cdot f(x)} = 7.05$$

The standard error of 7.05 from the curve, compared with that of 7.34 from the straight line, indicates that in both cases the amount of feed fed horses in a year can be estimated from the number of days they work in a year with a standard error of between 700 and 750 pounds. It also appears that the estimates made on the basis of the curvilinear relation are a little more reliable than those based on the linear relation.

The standard error of estimate can be used to indicate the probable reliability of a series of estimates of the values of the dependent variable when only the values of the independent variable are known, only where it is definitely known that the values of the new series are taken from exactly the same universe—the same set of conditions—as were the observations from which the relation was determined. In case they do not represent exactly the same conditions—as if, for example, they represent a different period of time[1]—then the standard error of estimate has meaning only with respect to the scatter of the residuals around the regression line *for the cases used in determining the relationship.* It measures (when adjusted) what the differences probably would have been in the universe from which the observations came, but does not give more than a clue or a possible indication as to what the differences may be when the same relations are applied to data from new or different conditions.

The standard error of estimate is necessarily stated in exactly the same kind of units that the original dependent variable is stated in. Where the dependent variable is stated in feet, as in the automobile problem, the standard error of estimate will be in feet; where it is in percentage points, as in the wheat problem, the standard error will be in percentage points; and where it is in logarithms, as in Table 27, the standard error will be in logarithms. Thus for Table 27, the standard error of estimate is the logarithm 0.038. That means that the logarithm of the estimates is apt to agree with the logarithm of the true values to within ±0.038, two-thirds of the time. With an estimated logarithm of 1.00, the logarithm of the true value would then be between 0.962 and 1.038, two-thirds of the time. Taking antilogarithms, this gives values of 9.16 and 10.91, or between 9.1 per cent above and 8.4 per cent below the value 10. Since a given logarithmic difference always means the same percentage difference, no matter how large or how small the base to which it is applied, when the standard error is thus stated in logarithms it indicates the range within which the estimates may be expected to be reliable, not as absolute quantities such as pounds of grain, but as percentages. Using absolute differences, the estimate might be expected to be right within 100 pounds, no matter whether the quantity fed was estimated at 1,000 pounds or 4,000 pounds; whereas using logarithms, if the estimate was expected to be right within 100 pounds for an estimate of 4,000 pounds, it would be expected to be right within 25 pounds for an estimate of 1,000 pounds.

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[1] See Chapter 2, page 14, for the other conditions assumed before error formulas apply exactly.

The *standard error of estimate* is thus computed from the standard deviation of the residuals for the cases on which the relation is based; it may be used to indicate the probable reliance to be placed in estimates for new observations, where the new cases are drawn from the same conditions as the first set of observations; but its exact interpretation differs with the particular units in which the values of the independent variable are expressed.

The Relative Importance of the Relationship—Correlation.—In certain problems it might be found that every bit of variation in one variable could be explained, or accounted for, by associated differences in the value of an accompanying variable. Thus all the variation in the volume of a cube can be explained by the corresponding difference in the length of one side. No other variable is needed to account for the volume of the cube. If we know what the length of the side is, we can estimate accurately what the volume will be. All of the variation in volume can therefore be said to be explained, or accounted for, by the known relation to the length of the side.

In most problems with which the statistician has to deal, however, all of the variation cannot be explained by the relation to another variable, and residual variation is left over. As has just been pointed out, this residual variation can be measured and used as an indication as to the errors in estimate which are likely to be made in estimating for further observations.

It is obvious that if no relation has been found, the independent variable considered does not explain any of the observed variation in the dependent variable, and so none of the variation can be explained as due to, or associated with, the independent variable. If, as in the case of the cube, the estimates all agree exactly with the actual values, there are no residual elements, and the variation is perfectly explained. But between these two extremes lie the cases of partial explanation, where a portion of the variation can be explained by the independent variable considered, and a portion cannot. In the automobile case, part of the variation in stopping distance, but not all, was associated with the speed; in the wheat case, part of the variation in protein content, but not all, could be estimated from variations in the proportion of vitreous kernels; and in the horse feed case, part of the variation in feed fed, but not all, could be accounted for by variations in number of days worked. In many problems it is of interest to determine what proportion of the variation in the dependent variable can be explained by the particular independent variable considered, according to the relation observed.

Measurement of the relative importance of the relation between two variables calls for a different type of statistical constant than the standard error of estimate. The standard error of estimate simply indicates the size of the residuals without regard to the amount of variation in the dependent variable as first observed. If the standard error of estimate for a cotton yield problem, for example, were 50 pounds, that would be the standard error no matter whether the yield of cotton in the original cases varied between only 200 and 400 pounds, or varied between 50 and 1,200. If the yields varied only between 200 and 400 pounds, and the standard error was 50, practically all of the variation in the original yields would still be left in the residuals; whereas if the yields varied between 200 and 1,200, and the standard error was 50, only a very small portion of the original variation would be left in the residuals. Yet the standard error of estimate would be of the same size in either case.

What is needed to show the relative importance of the relationship is some measure which shows what *proportion* of the original variation has been accounted for. The amount of the variation in the series of estimated (Y') values shows *how much* variation has been accounted for. All that need be done is to compare that variation with the variation in the original series to determine what proportion of the variation has been explained.

The standard deviation may be employed for the purpose of measuring the amount of variation. The actual values, Y, shown in Table 28, have a standard deviation of 7.92. The values estimated from the linear regression equation, Y', have a smaller standard deviation, 3.47. If we determine how large the latter is compared to the former, we get $\sigma_{y'}/\sigma_y = 3.47/7.92$, or 0.44. This is then a measure of the importance of relationship between the two variables—or the amount of *correlation*, as it is termed—according to the particular type of curve for which the relationship was determined.

Linear Relations—Coefficient of Correlation.—Where the relationship between the two variables is found or assumed to be a straight line, the value of $\sigma_{y'}/\sigma_y$ is termed the *coefficient of correlation*. The symbol r is used to represent it. When values of Y are estimated from values of X according to a straight line equation, then the proportion of the variation in Y which is so accounted for is indicated by the notation r_{yx}, which is read "the coefficient of correlation between Y and X."

The coefficient of correlation may therefore be defined

$$r_{yx} = \frac{\sigma_{y'}}{\sigma_y} \qquad (21)$$

The method of computing the coefficient of correlation which has just been shown, demonstrates that the coefficient is simply a measure of how large the variation in the estimated values is, in proportion to the variation in the original values. The coefficient of correlation thus measures the *proportion* of the variation in one variable which is associated with another variable, and therefore is a measure of the relative importance of the concomitance of variation in the two factors.

Curvilinear Relations—Index of Correlation.—In case the relation has been determined as a curvilinear function instead of a straight line, then the ratio $\sigma_{y''}/\sigma_y$ is termed the *index* of *correlation*, and is represented by the symbol ρ_{yx}.

The index of correlation may therefore be defined as

$$\rho_{yx} = \frac{\sigma_{y''}}{\sigma_y} \tag{22}$$

Computing the index of correlation for the horse feed case, $\sigma_{y''}/\sigma_y = 3.86/7.92 = 0.49$. From this figure, it would appear that the correlation is definitely higher for the curve than for the straight line.

It should be noted that in the case of straight-line relations, if the line has a positive slope, so that as X increases the values of Y' (the estimated values of Y) increase, then the correlation is said to be *positive*, and a plus sign is affixed to the correlation coefficient. Similarly, if the line has a negative slope, so that as the values of X (the independent variable) are larger, the values of Y' (the estimated values for the dependent variable) become smaller, then the correlation is said to be negative, and a minus sign is affixed to the correlation coefficient. In the case of the correlation index, the curve may be positive in one portion and negative in another, so no sign is used, and reference to the curve is necessary to indicate the nature of the relationship.

In a case where the observed relation explains *all* the variation in the dependent variable, the estimated values will be identical with the actual values. The standard deviation of Y' will therefore be exactly as large as the standard deviation of Y, and the ratio $\sigma_{y'}/\sigma_y$ will equal 1.0. This is termed *perfect correlation,* and is indicated when $\rho = 1.0$, or when $r = +1.0$ or -1.0.

At the other extreme of no relation, no variation can be accounted for by the particular independent variable considered, and the estimated values Y' are therefore all the same, being merely the average of Y. In that case the standard deviation of the estimated values is zero, and the ratio $\sigma_{y'}/\sigma_y = 0/\sigma_y = 0$. The case of complete absence of correlation, therefore, is indicated by values of 0 for either r or ρ.

The possible values of the coefficient of correlation therefore range from 0 to $+1.0$ or to -1.0; whereas the values for the index of correlation range from 0 to 1.0. Since most problems with which the investigator has to deal involve cases that are intermediate, where there is some correlation but not perfect correlation, it is these intermediate cases which are of most importance. The precise significance of different values of r and ρ will next be considered.

Where both X and Y are assumed to be built up of simple elements of equal variability all of which are present in Y but some of which are lacking in X, it can be proved mathematically that r^2 measures that proportion of all the elements in Y which are also present in X. For that reason in cases where the dependent variable is known to be causally related to the independent variable, r^2 may be called the coefficient of determination. It may be said to measure the per cent to which the variance in Y is determined by X, since it measures that proportion of all the elements of variance in Y which are also present in X.[2] Where elements are present in each variable which do not occur in the other, then the coefficient of determination is the product of these joint proportions. That is, if 2/3 of the elements in X are the same as 2/3 of the elements in Y, then the coefficient of determination will be equal to 4/9.

Although the coefficient of correlation was the earliest measure used, it can be seen that it may be misinterpreted. Thus if half the variance in Y is directly due to X, the coefficient of correlation would be 0.707 $(=\sqrt{\tfrac{1}{2}})$. Yet the coefficient of alienation[3] is also 0.707. If instead 376 the coefficient of determination is used, when we know that that is 0.50, we know at once that the coefficient of non-determination[3] is also 0.50; or if the determination is 0.60, the non-determination is 0.40. Since this is the most direct and unequivocal way of stating the proportion of the variance in the dependent factor which is associated with the independent factor, it may be used in preference to the other methods.

Where curvilinear relations have been used in determining the relationship, the term " index of determination " will be used to denote the value of ρ^2, thus retaining the same relation to the index of correlation that the coefficient of determination bears to r, the coefficient of correlation. When an expression is used such as " Forty per cent of the variance in yield is due to differences in rainfall," it will be understood that it is either the coefficient or the index of determination which is being stated.

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[2] See Note 4, of the technical appendix.

[3] See Note 5, of the technical appendix, for a fuller definition of these new terms.

Adjustments for Number of Observations.—Where the number of cases in the sample is not very large, both the coefficient or index of correlation, and the standard error of estimate, require certain adjustments before the values calculated from the sample, as given by equations (20) to (22), can be used to indicate the values which are most probably true for the universe from which that sample was drawn. Without these corrections, the observed coefficient or index of correlation tends to exceed the true correlation, while the observed standard error of estimate tends to fall below the true standard error.[4]

Denoting the corrected constants as $\bar{S}$, $\bar{r}$, and ρ, the correction formulas are as follows:

See R.A. Fisher 117.

$$\bar{S}^2_{y \cdot x} = S^2_{y \cdot x}\left(\frac{n-1}{n-2}\right);$$

hence

$$\bar{S}^2_{y \cdot x} = \frac{n\sigma^2_z}{n-2} \tag{23}$$

and for curvilinear correlation:

$$\bar{S}^2_{y \cdot f(x)} = S^2_{y \cdot f(x)}\left(\frac{n-1}{n-m}\right);$$

hence

$$\bar{S}^2_{y \cdot f(x)} = \frac{n\sigma^2_{z''}}{n-m} \tag{24}$$

For the correlation coefficient and index:

$$\bar{r}^2_{yx} = 1 - (1 - r^2_{yx})\left(\frac{n-1}{n-2}\right) \tag{25}$$

$$\bar{\rho}^2_{yx} = 1 - (1 - \rho^2_{yx})\left(\frac{n-1}{n-m}\right) \tag{26}$$

If the value to the right of the first "1−" in equation (25) or (26) exceeds unity, then 0 must be taken for the value $\bar{r}$ or $\bar{\rho}$.

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[4] The value of r calculated from a sample is derived from the standard deviation of the estimated values $\sigma_{y'}$ and the standard deviation of the dependent variable σ_y. It was noted in Chapter 2 that when standard deviations are computed from a small sample, they tend to fall below the true standard deviation of the universe, and this applies to σ_y. At the same time, $\sigma_{y'}$ is determined from a limited number of observations. It was already pointed out that a straight line would exactly fit any two observations with no residuals at all. When a straight line is fitted to ten observations, there are only eight "degrees of freedom" in determining the values a and b, as the "freedom" of two of these observations is used up in the determination. As a consequence of these conditions, the $\sigma_{y'}$ tends to be larger than it should be, and σ_y tends to be too small. Hence the quotient, $\sigma_{y'}/\sigma_y$ tends to be too large, on the average. Also, since $\sigma_{y'}$ tends to be too large, σ_z tends to be too small, and hence the observed standard error of estimate also needs correction.

$\bar{\sigma}_y^2 = \sigma_y^2 \cdot \frac{N}{N-1}$ N-1 allows for one parameter

$\sigma_{y'} = \sigma_{yc}$

$\sigma \;= S_y$

The corrected value $\bar{r}$, is the value which most probably exists in the universe, if the correlation is 0.80 or better—that is, the chances are even that the true correlation is either higher than $\bar{r}$ or lower than $\bar{r}$. If, however, the correlation is low—0.60 or less—then $\bar{r}$ is a somewhat more conservative estimate of the true correlation—that is, the chances are between 60 and 70 out of 100 that the true correlation is really as high as the adjusted value, $\bar{r}$. A corresponding explanation holds true for the other corrected values; that is, the odds are at least even that the true standard error is no higher than the observed standard error, as corrected. In equations (23) to (26), n stands for the number of observations. In equations (24) and (26), m stands for the number of constants in the regression equation, such as a, b, c, etc. In the case of a parabola of the second order (type a), m would be 3; for a cubic parabola (type f), it would be 4. Where a free-hand curve has been used, it is necessary to estimate how many constants would be needed to represent the curve mathematically.

The standard error of estimate in estimating grain fed the horses by the linear equation, adjusting the standard deviation of the residuals by equation (23), works out as follows:

$$\bar{S}^2_{y \cdot x} = \frac{n\sigma^2_z}{n-2}$$

$$\bar{S}^2_{y \cdot x} = \frac{18(7.13^2)}{18-2} = 57.19$$

$$\bar{S}_{y \cdot x} = 7.56$$

The new value indicates that the errors in estimating grain from days worked, under the conditions previously stated, will tend to run slightly larger than was indicated by the residuals for the cases included in the study, as tabulated in Table 29.

When the standard deviation for the curvilinear function is calculated by equation (24), a different result than before appears. If it is assumed that the regression curve used could have been represented mathematically by an equation with three constants (such as a parabola) then the correction works out:

$$\bar{S}^2_{y \cdot f(x)} = \frac{n\sigma^2_{z''}}{n-m}$$

$$\bar{S}^2_{y \cdot f(x)} = \frac{18(6.85^2)}{18-3} = 56.31$$

$$\bar{S}_{y \cdot f(x)} = 7.50$$

The adjusted standard error of estimate for the curvilinear relation, 7.50, is only barely smaller than that for the linear equation, 7.56. This indicates that when estimates are made for new samples from the same universe, the straight line is likely to give about as reliable results (within the range shown by the original sample) as is the regression curve. Not unless the adjusted standard error for the curve is definitely smaller than for the straight line can the curvilinear regression be relied upon to improve the accuracy of estimate.

Applying the correction to the value of r_{yx} previously computed for the horse problem, the correlation of grain fed with number of days worked is found to be:

$$\bar{r}^2_{yx} = 1 - \frac{[1 - (0.44)^2]\,(18 - 1)}{18 - 2} = 0.1432$$

$$\bar{r}_{yx} = 0.38$$

The index of correlation is even more likely to be spuriously high when based on a small number of cases than is the coefficient of correlation, and is even more in need of the adjustment, indicated by equation (26).[5]

Computing the index of correlation for the horse feed problem, with the corrections shown in equation (26):

$$\bar{\rho}^2_{yx} = 1 - \frac{(18 - 1)\left(1 - \frac{3.86^2}{7.92^2}\right)}{18 - 3} = 0.1389$$

$$\bar{\rho}_{yx} = 0.37.$$

After adjusting, we find that in this case the index of correlation is about as large as the coefficient, agreeing with the conclusion shown by the two standard errors of estimate. Just as with the standard errors, so it is with the correlation—not unless the index of correlation is still definitely higher than the coefficient, after they have been adjusted by formulas (25) and (26), can it be said that there is definite indication of curvilinear correlation rather than of linear.[6]

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[5] The adjusted index of correlation $\bar{\rho}$ has the same interpretation as the adjusted coefficient of correlation—it is the correlation which probably exists in the universe from which the sample is drawn, for odds of at least one chance out of two.

Just as the a and b of the linear equation eliminate two degrees of freedom, a curve representing three constants (or more) can be passed exactly through three observations (or more) and so may eliminate three (or more) degrees of freedom. There is therefore even more tendency for ρ to be spuriously high than for r, and the correction is even more needed.

It should be noted that in any case the adjustment to r or ρ is small compared with its own standard error—that is, the true value in the universe may be different from the value observed in the sample, by a margin much larger than the difference between the observed value and the adjusted value. Chapter 18 discusses methods of estimating the probable range. Even so, the apparent value from a sample always tends to have the bias mentioned; and it is worth eliminating this bias as far as possible, even if the adjusted value is still subject to a considerable standard error of its own.

Summary.—This chapter has pointed out that the closeness of relation between two variables may be measured either by the absolute closeness with which values of one may be estimated from known values of the other, or on the basis of the proportion of the variation in one which can be explained by, or estimated from, the accompanying values of the other. The absolute accuracy of estimate is measured by the standard error of estimate, which indicates the reliability of values of the dependent variable estimated from observed values of the independent value.

The relative closeness of the relation is best measured by the coefficient of determination, in the case of linear relationship, or by the index of determination, in the case of curvilinear relationship. These measures show the proportion of the variance in the dependent variable which is associated with differences in the other variable. In the case of variables causally related, they measure the proportion of the variance in one which can be said to be *due to* the other.

The best methods of computing the various measures of correlation will be shown in the next chapter; the methods used in this chapter are designed rather to show the significance of the measures themselves.

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[6] See Figure *F* of Appendix 3 for a ready method of computing adjusted coefficients or indexes of correlation from the unadjusted values.

CHAPTER 8

PRACTICAL METHODS FOR WORKING TWO-VARIABLE CORRELATION PROBLEMS

Terms to be Used.—The preceding discussion has developed the means by which one variable may be estimated from another. Simple correlation involves only the means for making such estimates, and for measuring how closely those estimates conform to, and account for, the original variation in the variable which is being estimated.

The *regression line* is used, in statistical terminology, to designate the straight line used to estimate one variable from another by means of the equation

$$Y = a + bX$$

This equation is termed the *regression* equation; and the coefficient b, which shows how many units (or fractional parts) Y changes for each unit change in X, is termed the *coefficient* of *regression.*

Where a curvilinear function has been determined, either by the use of an equation or by graphic methods, the corresponding curve is similarly designated as the *regression curve.* Either the mathematical equation, or if none has been computed, the expression

$$Y = a + f(x),$$

where the symbol $f(x)$ stands for the relation shown by the graphic curve, is termed the *regression equation.*

The coefficient of correlation and the index of correlation have both been defined as the (adjusted) ratio of the standard deviation of the estimated values of Y to the standard deviation of the actual values, whereas the standard error of estimate has been defined as the standard deviation of the residuals from the estimates so made. In the case of linear relations, however, the coefficient of correlation and the standard error of estimate can both be computed directly from almost the same operations as were employed in computing the constants of the regression equation. This will be illustrated by the practical example which follows.

Working Out a Linear Correlation.—As was illustrated in Chapter 5, pages 55 to 58, the values for a and b of the regression equation can be determined for any two variables, X and Y, between which it may be desired to determine the relation, by working out the values, M_x, M_y, ΣX^2 and $\Sigma(XY)$, and then substituting them in the appropriate equations. In order to compute directly the coefficient of correlation, r_{xy}, and the standard error of estimate, S_{yx}, it is only necessary to compute in addition the value ΣY^2, and substitute it in appropriate formulas. The data given in Table 31 illustrate the necessary operations.

TABLE 31

Computing the Values Needed to Determine Linear Regression and Correlation Coefficients

	Irrigation water applied per acre* (X)	Yield of Pima cotton per acre* (Y)	X^2	XY	Y^2
	Feet	*Units of ten pounds*			
	1.8	26	3.24	46.8	676
	1.9	37	3.61	70.3	1,369
	2.5	45	6.25	112.5	2,025
	1.4	16	1.96	22.4	256
	1.3	9	1.69	11.7	81
N=14	2.1	44	4.41	92.4	1,936
	2.3	38	5.29	87.4	1,444
	1.5	28	2.25	42.0	784
	1.5	23	2.25	34.5	529
	1.2	18	1.44	21.6	324
	1.3	22	1.69	28.6	484
	1.8	18	3.24	32.4	324
	3.5	40	12.25	140.0	1,600
	3.5	65	12.25	227.5	4,225
Total.	27.6	429	61.82	970.1	16,057
Mean.	1.97	30.64 30.643			

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* From James C. Muir and G. E. P. Smith, The Use and Duty of Water in the Salt River Valley, University of Arizona, Agricultural Experiment Station Bulletin 120, 1927. All the plots were on the same type of soil, Maricopa sandy loam.

The computations shown in this table—squaring both X and Y, calculating the product XY, summing both X, Y, and the three columns of extensions, and dividing the first two sums by the number of cases to give the mean of X and Y—provide all the basic data necessary.[1] The values a and b for the regression equation may next be computed by substituting these extensions in equations (9) and (10), which were used previously in Chapter 5, page 57.

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[1] Where the number of cases to be handled is large, various short-cuts may be used

$$(9) \qquad b_{yx} = \frac{\Sigma(XY) - nM_xM_y}{\Sigma(X^2) - n(M_x)^2} = \frac{970.1 - 14(1.97)(30.64)}{61.82 - 14(1.97^2)}$$

$$= \frac{125.050}{7.4874} = \boxed{16.701}$$

additional decim
points give 16.79

$$(10) \qquad a = M_y - bM_x = 30.64 - 16.701(1.97) = -2.261$$

30.143 - 16.797(1.97143) = -2.471

The *regression line*, $Y = a + bX$, therefore is for this case

$$Y = -2.261 + 16.701X$$

The unadjusted coefficient of correlation, r_{xy}, may now be computed from the following new formula:

$$r_{xy} = \frac{\Sigma(XY) - nM_xM_y}{\sqrt{[\Sigma(X^2) - nM^2_x][\Sigma(Y^2) - nM^2_y]}} \qquad (27)$$

$$= \frac{970.1 - 14(1.97)(30.64)}{\sqrt{[61.82 - 14(1.97)^2][16{,}057 - 14(30.64)^2]}} = 0.847$$

It should be noticed that the numerator of this fraction is the same as that in the equation for b, and that half of the denominator is the same, except that it is under the radical sign.

Comparison of equations (9) and (27) with equation (5) for the standard deviation

$$\sigma_x = \sqrt{\frac{\Sigma(X^2) - nM^2_x}{n}}$$

$\sigma_y = 14.42$
$s_y = 7.67$

shows that they may be written more simply

$$b_{yx} = \frac{\Sigma(XY) - nM_xM_y}{n\sigma^2_x} \quad \text{or} \quad = \frac{\Sigma(xy)}{n\sigma^2_x}$$

$$r_{xy} = \frac{\Sigma(XY) - nM_xM_y}{n\sigma_x\sigma_y} = \frac{\Sigma(xy)}{n\sigma_x\sigma_y}$$

The second form, in each case, uses the notation $\Sigma(xy)$ for $\Sigma(XY) - n(M_xM_y)$ as discussed on page 57.[2] The forms shown in equations (9), (10), and (27), however, are the ones ordinarily used in actual computation, and should be kept clearly in mind.

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to reduce the volume of computation required in computing the sums of extensions ΣX^2, ΣXY, and ΣY^2. The use of these short-cuts is developed in the Appendix, pages 353 to 357.

[2] The value $\Sigma(xy)$ is sometimes called the *product moment*.

Solving simultaneous equation

970.1 = 27.6a + 61.82 b
429 = 14a + 27.6 b
a = -2.446
b = 16.785

Once r_{xy} has been computed, the value adjusted for the number of cases can then be obtained by equation (25),

$$\bar{r}^2_{xy} = 1 - (1 - r^2_{xy})\left(\frac{n-1}{n-2}\right)$$

For the present problem, that becomes

$$\bar{r}^2_{xy} = 1 - \frac{[1 - (0.847)^2](14 - 1)}{14 - 2} = 0.6939$$

$$\bar{r}_{xy} = 0.833$$

Knowing $\bar{r}_{xy}$, the standard error of estimate may next be computed by the following equation:

$$\bar{S}_{yx} = \sqrt{\frac{\Sigma(Y^2) - n(M_y)^2}{n-1}(1 - \bar{r}^2_{xy})} \qquad (28)$$

$$= \sqrt{\frac{16{,}057 - 14(30.64^2)}{13}[1 - (0.833)^2]}$$

$$= \sqrt{68.62} = 8.28$$

Since this equation includes $\bar{r}_{xy}$, already adjusted for the number of observations, no further adjustment is necessary. The standard error computed by equation (28) is identical with that obtained by equation (23), given in the previous chapter.

It should be noted that though $r_{xy} = r_{yx}$, b_{xy} is *not* the same as b_{yx}. The former regression, showing the change in X for each unit change in Y (that is, regarding the dependent factor as the independent factor instead), is obtained by modifying equation (9) to the following form:[3]

$$b_{xy} = \frac{\Sigma(XY) - nM_xM_y}{\Sigma(Y^2) - n(M_y)^2}.$$

Equation (28) completes the computation of all the values needed, except the coefficient of determination, d_{xy}, which is simply r^2_{xy}. That is:

$$\bar{d}_{xy} = \bar{r}^2_{xy} = (0.833)^2 = 0.694$$

Interpreting the Results of a Linear Correlation.—The next step is to take the several constants which have been computed and see what they mean.

The coefficient of regression of Y on X, $b_{yx} = 16.70$, shows that on the average the acre yield of cotton increases 16.7 ten-pound units, or 167 pounds, for each additional acre-foot of water applied. The constant a shows that with no water applied, a yield of -2.26 ten-pound units, −22.6 pounds, or less than no cotton at all, might be expected. Since these results are based on observations extending from 1.2 acre-feet of water to 3.5, the relations shown by the regression line do not necessarily hold beyond those limits, and it is not certain what the yield would be—extrapolating the regression line to that point is only a guess.

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[3] When the correlation is perfect, so that $r_{xy} = 1$, then the two regression coefficients will have the definite relation $b_{yx} = 1/b_{xy}$. Under these conditions the regression lines will be identical, no matter which variable is regarded as the independent variable and which as the dependent.

The regression equation

$$Y = -\ 2.26 + 16.7(X),$$

or

$$\text{Yield} = -\ 22.6 + 167\ (\text{feet of water}),$$

then gives the yields of cotton estimated as most likely to be obtained from the quantity of water applied within the limits from 1.2 to 3.5 feet. Figure 22 shows how these estimated values, along the regression line, compare with the actual yields observed.

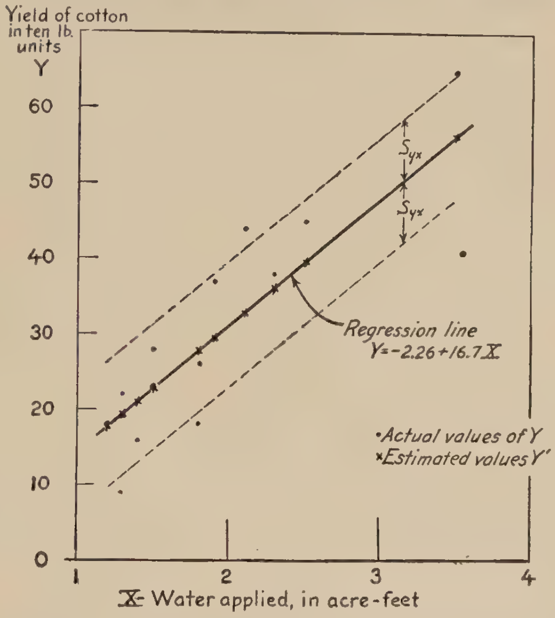

FIG. 22.—Relation of yield of cotton to irrigation water applied; estimated yields from a linear regression and zone of probable yields indicated by the standard error of estimate.

The standard error of estimate, 8.28 ten-pound units or 82.8 pounds, shows that the (adjusted) standard deviation of the differences between the actual and the estimated values is 82.8 pounds of cotton. Two lines have been drawn in Figure 22, at 82.8 pounds above and below the regression line. It will be seen that of the 14 cases, 9 fell between these two lines, or in the zone within one standard-error on either side of the regression line.

The coefficient of correlation, $\bar{r}_{xy} = 0.83$, and the coefficient of determination, $d_{xy} = 0.69$, show that about 69 per cent of the variance in the yield of this crop in this area, on the farms from which these records were obtained, could be accounted for by the differences in the quantity of water used in irrigation. Since this leaves only

31 per cent of the variance to be accounted for by all other factors, it would appear that the quantity of water applied (or other factors associated with it) was the most important factor which was associated with the yield of cotton on these farms and on this type of soil.

The fact that 69 per cent of the variance in yield can be explained by corresponding differences in the quantity of water applied does not in itself mean that the differences in irrigation *caused* the differences in yield. For example, it might be possible that the quantity of water applied was regulated to conform to the fertility of the land, and that the differences in yield were really due to the differences in fertility. The statistical measure merely tells how closely the variance in one variable was associated with variance in the other; whether that association is due to, or can be taken as evidence of, cause-and-effect relation is another matter, and is outside of the scope of the statistical analysis. (For more extended discussion of this point, see the last two chapters of this book.)

MP.

Working Out a Curvilinear Correlation.—The next step is to consider whether the straight line is adequate to describe the way that the yield increases as more water is applied, or whether a curve had better be employed. (This step can be taken before any of the linear results are worked out, and if a curve is decided on, the previous work can be skipped entirely, if desired.)

Examining Figure 22 more closely, we see that, in the range up to 1.8 acre-feet of water, the actual yields lie below the regression line four times, and above four times; in the range from 1.9 to 3 acre-feet, the actual yields lie above in all four observations; and above 3 acre-feet the one yield below the line is much farther below than is the one above. These facts suggest that a convex curve, giving lower estimated yields than the straight line for the lowest and highest applications of water, and higher estimated yields for the intermediate applications, would more accurately represent the relations in this case. (The number of observations is far too low to serve as a very accurate indication of the shape of the curve, but it will serve at least as a simple illustration of the way the whole problem may be worked through.)

The first step is to group the observations according to the value of X (the quantity of water) and average both X and Y, water and yield. In view of this small number of observations, rather large groups are taken; were more cases available, the groups might be made narrower.

TABLE 32

COMPUTATION OF GROUP-AVERAGES TO INDICATE REGRESSION CURVE—COTTON EXAMPLE

X (water) 1 to 1.4			X (water) 1.5 to 1.9		X (water) 2.0 to 2.9		X (water) 3.0 to 3.9	
	X	Y	X	Y	X	Y	X	Y
	1.4	16	1.8	26	2.5	45	3.5	40
	1.3	9	1.9	37	2.1	44	3.5	65
	1.2	18	1.5	28	2.3	38		
	1.3	22	1.5	23				
			1.8	18				
Sums.....	5.2	65	8.5	132	6.9	127	7.0	105
Means....	1.3	16.25	1.7	26.4	2.3	42.33	3.5	52.5

These averages are then plotted, as shown in Figure 23, an irregular line dotted in connecting them and as smooth a curve as possible drawn in free-hand through the averages and the broken line, just as discussed in pages 91 to 95, Chapter 6. This then gives the regression curve.

Next the estimated yields for each different application of water are read off from this curve, and the difference between the actual and the estimated yields is determined. These residuals are then squared to determine their standard deviation. In case the linear correlation has not been previously worked, the

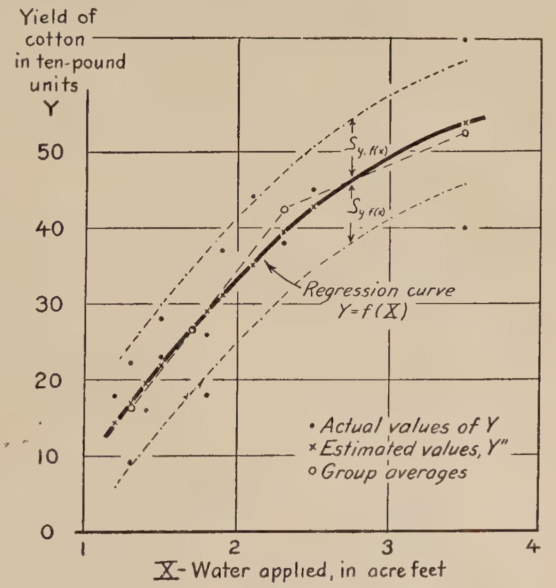

FIG. 23.—Relation of yield of cotton to irrigation water applied; estimated yields from a curvilinear regression; and zone of probable yields as indicated by the standard error of estimate.

yields, or Y values, are also squared as shown, so as to determine their standard deviation, and so give the basis for measuring the amount of correlation.

TABLE 33

COMPUTATION OF RESIDUALS AND STANDARD DEVIATION FOR CURVILINEAR REGRESSION—COTTON EXAMPLE

Water per acre, X	Yield, in ten-pound units, Y	Yield estimated from X, in ten-pound units, Y''	$Y - Y''$, (z'')	$(z'')^2$	Y^2
1.8	26	29.0	− 3.0	9.00	676
1.9	37	31.0	6.0	36.00	1,369
2.5	45	42.8	2.2	4.84	2,025
1.4	16	19.2	− 3.2	10.24	256
1.3	9	16.8	− 7.8	60.84	81
2.1	44	35.2	8.8	77.44	1,936
2.3	38	39.5	− 1.5	2.25	1,444
1.5	28	21.9	6.1	37.21	784
1.5	23	21.9	1.1	1.21	529
1.2	18	14.2	3.8	14.44	324
1.3	22	16.8	5.2	27.04	484
1.8	18	29.0	−11.0	121.00	324
3.5	40	54.0	−14.0	196.00	1,600
3.5	65	54.0	11.0	121.00	4,225
Sums.......	429	425.3	+ 3.7	718.51	16,057

The sum of the Y'' values is slightly smaller than the sum of the Y values, and the mean of the z'' values is therefore not exactly zero, but 0.264. That indicates that the curve shown in Figure 23 should be shifted up 0.264 units, or 2.64 pounds, to make the estimated and actual averages agree.[4] Representing this curve by $f(X)$, the regression equation for the curvilinear correlation may therefore be written:

$$Y = k + f(X)$$

$$Y = 2.64 + f(X)$$

---

[4] In problems with many observations, the sum of the Y values and of the Y'' values may be determined separately for the several different portions of the curve, to see if its position should be shifted in one portion and not in another. This process cannot be carried too far, however, for if the divisions are made too small the effect will be to make the curve pass through each successive group average, without smoothing out the irregularities into a continuous function.

The values at the foot of Table 33 now give the constants necessary to measure the closeness of the correlation. First the standard deviations of Y and of z'' are computed, using the formula

$$\sigma_y = \sqrt{\frac{\Sigma Y^2 - n(M^2_y)}{n}} = 14.44$$

$$\sigma_{z''} = \sqrt{\frac{\Sigma(z'')^2 - n(M^2_{z''})}{n}} = \sqrt{\frac{718.51 - 14(0.264^2)}{14}} = 7.16$$

Then, by equation (24),

$$\bar{S}^2_{y \cdot f(x)} = \frac{n\sigma^2_{z''}}{n - m} = \frac{14(7.16)^2}{14 - 3} = 65.23$$

$$\bar{S}_{y \cdot f(x)} = 8.07$$

Here 3 is used for the value of m, since it is judged that a parabolic equation of type (a), with 3 constants, would be adequate to reproduce the free-hand curve.

The standard error of estimate for the graphic regression curve is thus 8.07 ten-pound units, or 80.7 pounds. This is 2.1 pounds smaller than the corresponding value in the case of the linear correlation, indicating how much more closely the curve fits the data than does the straight line, even after allowing for its greater flexibility. In Figure 23 two dotted lines have been drawn in, each 80.7 pounds away from the regression curve, indicating the zone of estimate within which approximately 2/3 of the cases fall (10 out of 14 in this case) and within which 2/3 of the actual yields may be expected to fall in case new estimates of yield are made from the water applied for additional cases drawn from the same universe. (Note the discussion on pages 108 to 110 of Chapter 6, on the reliability of such estimates.)

The index of correlation, $\bar{\rho}_{yx}$, may next be computed by substituting the two standard deviations in formula (29):

$$\bar{\rho}^2_{yx} = 1 - \left(\frac{\sigma^2_{z''}}{\sigma^2_y}\right)\left(\frac{n - 1}{n - m}\right) \tag{29}$$

This formula includes the corrections for the number of variables and constants. It should always be used in calculating the index of correlation where the curve has been determined free-hand, as in this case, since it gives a more accurate measure of the correlation than does equation (22), shown previously.

Where the equation of the curve has been determined by mathematical means, the standard error of estimate and the index of correlation may be computed without working out the estimates and residuals for each of the individual cases. These methods will be described subsequently.[5]

In the example given, the index of correlation works out

$$\bar{\rho}^2_{yx} = 1 - \left[\frac{(7.16)^2}{(14.44)^2}\right]\left[\frac{14 - 1}{14 - 3}\right] = 1 - 0.2905 = 0.7095$$

$$\bar{\rho}_{yx} = 0.842$$

Since the index of determination is simply $\bar{\rho}^2_{xy}$, it is 71.0 per cent. Comparing these results with those obtained by linear correlation the index of determination of 71.0 per cent compares with the coefficient of determination of 69.4 per cent. Apparently taking into account the curvilinear nature of the relations has increased the proportion of the variance in yield accounted for by differences in water application by 1.6 per cent of the total variance in the yield. (Only the measures of determination can be directly compared in this way. If the coefficient of correlation, 0.833, were subtracted from the index of correlation, 0.842, that would give an incorrect idea of the importance of taking account of the curvilinear nature of the relation.)

Interpreting the Results of Curvilinear Correlation.—The index of determination and the accompanying standard error of estimate have been interpreted for the curve in much the same manner as were the coefficient of determination and the standard error of estimate for the straight line. In the case of the regression curve itself, however, a somewhat different method of presentation may be best, since a mathematical equation expressing the relation has not been computed.

The regression curve just worked out for the cotton problem, for example, may be presented either as a curve showing graphically the yield to be expected for various applications of water, as is illustrated in Figure 23, or as a table showing the same thing, as in Table 34. In both cases the constant which has been determined from the average of z'' is added to the values read from the curve in Figure 23, $f\ (X)$, so as to give the final estimates which would be made by taking into account this slight shift in the position of the curve.

Similar presentation could be given the regression line in cases of linear correlation, if desired, but then the chart would show only a straight line, and the table would show exactly the same changes in the dependent variable for each successive uniform change in the independent variable. In preparing the table, the relation is shown only for that range of water application within which the bulk of the observations fall. Similarly, only this range should be shown by the solid line in the chart; a dotted line might be used to indicate the relations beyond that up to the extremes observed. Neither the regression line nor curve should, ordinarily, be carried beyond the limits of the observations on which it was based. (Note in this regard the discussions on pages 108 to 110, Chapter 6, and in Chapter 18.)

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[5] See page 315, Chapter 21.

TABLE 34

Yield of Pima Cotton, with Different Applications of Irrigation Water, on Maricopa Sandy Loam Soils in the Salt River Valley, Arizona, in 1913, 1914, and 1915

Irrigation water applied	Average yield of cotton lint
Acre-feet	*Pounds per acre*
1.25	156
1.50	222
1.75	283
2.00	335
2.25	385
2.50	431

Summary.—This chapter has illustrated the way in which correlation analysis may be applied to a specific problem, the manner in which linear and curvilinear regressions may be determined most simply, and the way in which they may be interpreted. In addition, the simplest manner of computing the standard error of estimate and the coefficient and the index of correlation have been illustrated, and their significance has been briefly discussed.

CHAPTER 9

THREE MEASURES OF CORRELATION—THE MEANING AND USE FOR EACH

So many different statistical coefficients have been introduced in the discussion of correlation that there may be some confusion between them as to the meaning and use of the different coefficients. Particularly in linear correlation, there are three constants which summarize nearly all that a correlation analysis reveals.

First, the standard error of estimate shows how nearly the estimated values agree with the values actually observed for the variable being estimated, and indicates about how near to the true values subsequent estimates made from the same regression equation and under the same conditions are likely to fall. This coefficient is stated in the same units as the original dependent variable, and its size can be compared directly with those values.

Second, the coefficient of determination (r^2) shows what proportion of the variance in the values of the dependent variable can be explained by, or estimated from, the concomitant variation in the values of the independent variable. Since this coefficient is a ratio, it is a ''pure number''; that is, it is an arbitrary mathematical measure, whose values fall within a certain limited range, and it can be compared only with other constants like itself, derived from similar problems.

Finally, the coefficient of regression measures the slope of the regression line; that is, it shows the average number of units increase or decrease in the dependent variable which occur with each increase of a specified unit in the independent variable. Its exact size thus depends not only on the relation between the variables, but on the units in which each is stated. It can be reduced to another form, however, by stating each of the variables in units of their own individual standard deviation. In this form it has been termed β or the ''beta'' coefficient.[1] The relation between beta and the coefficient of regression may be indicated by stating the regression equation in both ways:

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[1] See Truman Kelley, Statistical Method, p. 282.

$$Y = a + b_{yx}X$$

$$\frac{Y}{\sigma_y} = a' + \beta_{yx}\left(\frac{X}{\sigma_x}\right)$$

$$\beta_{yx} = b_{yx}\left(\frac{\sigma_x}{\sigma_y}\right)$$

$$a' = \frac{M_y}{\sigma_y} + \beta_{yx}\frac{M_x}{\sigma_x}$$

Stated in this way, β for the cotton yield problem is 0.845. That is, for each increase of one standard deviation (0.73 acre-foot of water) in X, the yield of cotton increased 0.845 of one standard deviation. Since the standard deviation of Y was 144.3 pounds, that is equal to 121.9 pounds of cotton for each 0.73 acre-foot of water. This is at the rate of 167 pounds of cotton for each foot of water, which is the same thing as was shown by the coefficient of regression. However, for comparisons between problems where the standard deviations are much different, the "beta" coefficient may have value. It is evident that in simple correlation the value of beta is the same as that of r.

Relation of the Different Coefficients to Each Other.—Even though each of the three coefficients measures certain aspects of the relation between variables, it does not follow that all three coefficients will vary together, or that a problem which shows a high coefficient of determination will also show a high regression coefficient or a low standard error of estimate. That is because they measure different aspects of the relation.

The particular usefulness of each of the three different groups of correlation measures is illustrated in Figure 24, which shows 3 sets of simple relationships, with hypothetical data.

Here the regression coefficient is smaller in A than B. In A an additional inch of rain causes an average increase of 2.5 bushels in yield, as compared with an increase of 3.1 bushels in B. But in case A, a considerable part of the variation in yield is apparently due to rainfall, as shown by the high correlation ($r = 0.83$) and the small size of the standard error of estimate (2.2 bushels); whereas in case B, factors other than rainfall apparently cause most of the differences in yield, as indicated by the lower correlation ($r = 0.71$) and the larger standard error of estimate (3.8 bushels). In terms of determination apparently about 69 per cent of the differences in yield are related to differences in rainfall in the first case, and only about 50 per cent in the second.

In comparison with *A* and *B*, case *C* has much less variable yields, ranging from only about 8 bushels to 12 bushels, compared with a range of 8 to 21 in case *A*, and 0 to 20 in case *B*. Only a small part (22 per cent) of the variation in yields is associated with rainfall differences, as indicated by the low correlation (0.47). An increase of 1 inch in rainfall apparently causes only 0.5 bushel increase in yield. Yet in spite of this low relation, it is possible to estimate yields more accurately, given the rainfall, in this case than in either of the other two, as is shown by the standard error of estimate of 1.15 bushels, as compared to 2.2 bushels for *A* and 3.1 for *B*. The original variation in yields is so slight in case *C* that even the small relation shown to rainfall is enough to make it possible to estimate yields more accurately than in either of the other cases.[2]

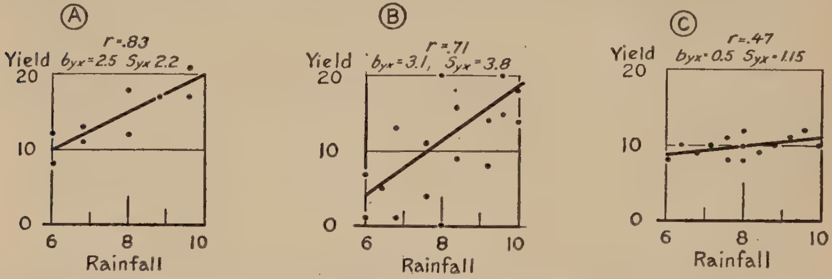

FIG. 24.—Hypothetical sets of data, illustrating three types of correlation coefficients.

These three cases illustrate the relative place of each of the three types of correlation measure. Case *B* shows the greatest change in yield for a given change in rainfall (the regression measure); case *A* shows the highest proportion of differences in yields accounted for by rainfall (the correlation or determination measure); and case *C* shows the greatest accuracy of estimate (the error of estimate measure). Which of these measures should have most attention in a particular investigation depends upon the phase of the investigation which is most important: the *amount* of change (regression); the *proportionate* importance (correlation); or the *accuracy* of *estimate* (standard error). All have their place, and none should be entirely overlooked or ignored.

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[2] In calculating the measures for these illustrative cases, the corrections for numbers of cases have been ignored, as they would not have affected the particular points these examples were set up to illustrate.

CHAPTER 10

DETERMINING THE WAY ONE VARIABLE CHANGES WHEN TWO OR MORE OTHER VARIABLES CHANGE

(1) By successive elimination

The Problem of Multiple Relations.—The relations studied up to this point have all been of the type where the differences in one variable were considered as due to, or associated with, the differences in one other variable. But in many types of problems the differences in one variable may be due to a number of other variables, all acting at the same time. Thus the differences in the yield of corn from year to year are the combined result of differences in rainfall, temperature, winds, and sunshine, month by month or even week by week through the growing season. The premiums or discounts at which different lots of wheat sell on the same day vary with the protein content, the weight per bushel, the amount of dockage or foreign matter, and the moisture content. The speed with which a motorist will react to a dangerous situation may vary with his keenness of sight, his speed of nervous reaction, his intelligence, and his familiarity with such situations. The price at which sugar sells at wholesale may depend upon the production of that season, the carryover from the previous season, the general level of prices, and the prosperity of consumers. The weight of a child will vary with its age, height, and sex. The volume of a given weight of gas varies with the temperature and the barometric pressure.

The physicist and the biologist use laboratory methods to deal with problems of compound or multiple relationship. Under laboratory conditions all the variables except the one whose effect is being studied may be held constant, and the effect determined of differences in the one remaining varying factor upon the dependent variable, while effects of differences in the other variables are thus eliminated. In the case of a gas, for example, the temperature may be held constant while the volume at different barometric pressures is determined experimentally, and then the pressure held constant while the volume at different temperatures is determined. For many of the problems with which the statistician has to deal, however, such laboratory controls cannot be used. Rainfall and temperature and sunshine vary constantly, and only their combined effect upon crop yields can be noted. Economic conditions are constantly shifting, and only the total result of all the factors in the existing situation can be measured at any time. And so on through many other types of multiple relations similar to those mentioned—the statistician has to deal with facts arising from the complex world about him, and frequently has but little opportunity to utilize laboratory checks or artificial controls.

The problem of multiple relations is illustrated by the data in Table 35. These represent 20 farms in one area, with varying crop acreages, dairy cows, and incomes. To determine from these records what income may be expected, on the average, with a given size of farm and with a given number of cows, it is necessary to estimate

TABLE 35

Acres, Number of Cows, and Incomes, for 20 Farms

Size of farm	Size of dairy	Income
Number of acres	*Number of cows*	*Dollars per year*
60	18	960
220	0	830
180	14	1260
80	6	610
120	1	590
100	9	900
170	6	820
110	12	880
160	7	860
230	2	760
70	17	1020
120	15	1080
240	7	960
160	0	700
90	12	800
110	16	1130
220	2	760
110	6	740
160	12	980
80	15	800

the effect of differences in the number of acres on income and also the effect of differences in the number of cows on income.

From these data it would seem that both the size of the farm and the size of the dairy herd influence farm income, to judge from dot-charts showing the relation of income to acres (Figure 25) and of income to number of cows (Figure 26). It appears from these charts that there may be a slight tendency for the farms with the larger acreage in crops to have larger incomes, and a rather marked tendency for the farms with the larger number of cows to have larger incomes.

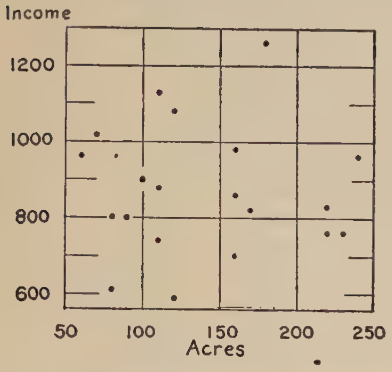

Fig. 25.—Correlation chart of acres and income on individual farms.

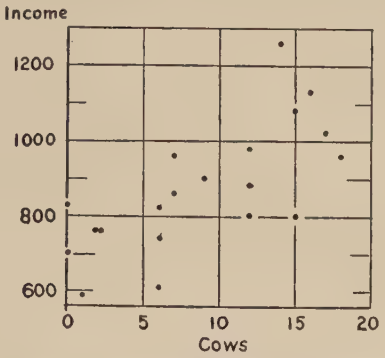

Fig. 26.—Correlation chart of number of cows and income on individual farms.

Analysis by Simple Averages Not Adequate.—The simple comparison alone, however, is not sufficient to tell exactly how incomes change with acres and with number of cows. That is because there is a marked relation between the size of the farms and the number of cows, as is illustrated in Figure 27. There is a definite tendency for the larger farms to have smaller dairy herds. As a result, the difference in incomes in Figure 25, which appeared to be due directly to differences in acreages, may be due in part to the differences in the sizes of the dairy herds on the farms with different acreages in crops. If we make groups of farms of 50 to 99 acres, 100 to 150 acres, and so on, and average

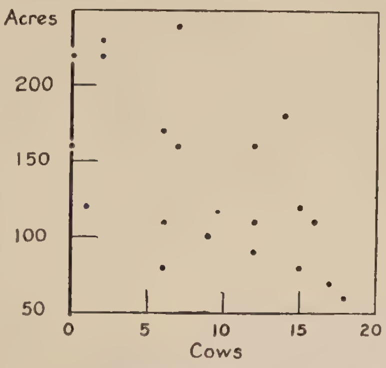

Fig. 27.—Correlation chart of number of cows and number of acres on individual farms.

the acres, cows, and income for each group, as is shown in Table 36, we find a marked difference in the number of cows from group to group, as well as in the number of acres and in the incomes.

TABLE 36

AVERAGE NUMBER OF COWS AND INCOME, FOR FARMS OF DIFFERENT SIZES

Size group	Number of farms	Average size	Average size of dairy	Average income
		Number of acres	*Number of cows*	*Number of dollars*
59–99 acres	5	76	13.6	838
100–149 acres	6	111	9.8	887
150–199 acres	5	166	7.8	924
200–249 acres	4	228	2.8	828

The farms of 50 to 99 acres, with an average size of 76 acres, have incomes which average $838; the farms of 150 to 199 acres, with an average size of 166 acres, show incomes which average $924. Is this difference in income due to the difference in size? Before this can be definitely answered we must consider that the two groups also differ in the average number of cows, with 13.6 in the first group, and only 7.8 in the second. So far, there is nothing to indicate whether the difference in income is due to the difference in the size of the farms or in the number of cows; we have shown that both vary from group to group, and that is all.

If, on the other hand, we should attempt to determine how far income varied with differences in the number of cows by classifying the records with respect to the number of cows, and averaging incomes, we would secure the result shown in Table 37.

TABLE 37

AVERAGE ACRES AND INCOME, FOR FARMS WITH DIFFERENT NUMBER OF COWS

Size of herd	Number of farms	Average size of dairy	Average size of farms	Average income
		Number of cows	*Number of acres*	*Number of dollars*
Under 5 cows	5	1.0	190	728
5–9 cows	6	6.8	143	815
10–14 cows	4	12.5	135	980
15 cows and over	5	16.2	88	998

Even though the income is higher on the farms with more cows, Table 37 does not indicate how much of that can be credited to the cows and how much to other factors. It is evident from the table that as the number of cows goes up, the number of acres goes down; are the differences in income associated with changes in number of cows, in number of acres, or in part with both?

Eliminating the Approximate Influence of One Variable.—What we need to know is how far income varies with size of farm, as between farms with the same number of cows; and how far income varies with the number of cows, as between farms of the same size as to acres. One way of determining this would be to adjust the income on each farm to eliminate the differences due to (or associated with) the number of cows, and then compare the adjusted incomes with the size of the farm to determine the effect of size on income. To start this process the effect of the number of cows upon incomes is needed. We can secure an approximate measure of this by determining the straight-line equation for estimating incomes from cows—approximate only, since the differences in the size of the farms are ignored at this point.

Determining the straight-line relation according to Chapter 5, we find that the relation between cows and income is given by the equation:

$$\text{Income} = \$694 + 20.11 \text{ (number of cows)}$$

According to this equation, farms with no cows averaged about $694 income, and these incomes increased $20.11 for each cow added, on the average. Knowing this relation, the incomes on the several farms can be adjusted by deducting that part of the income which would be assumed due to the cows, according to this average relation.

Table 38 illustrates the process of adjusting the incomes to a no-cow basis, by subtracting out this approximate effect of cows on incomes. The next step is to see what the relation is between the acres in the farm and these adjusted incomes. Plotting both on a dot-chart, Figure 28 shows this relation graphically. Comparing this figure with Figure 25, where the relation between the acres and the unadjusted incomes was plotted, it is seen that the relation is much closer and more definite for the adjusted incomes than for the unadjusted incomes. This is only natural; now that the marked relation of number of cows to income has been removed, even if only approximately, the underlying relation of size to income can be more clearly seen.

TABLE 38

Adjusting Farm Incomes for Differences in Number of Cows

Size of farm	Size of dairy	Income	Income assumed due to cows	Income adjusted to no-cow basis
Number of acres	*Number of cows*	*Dollars*	*Number of dollars*	*Number of dollars*
60	18	960	362	598
220	0	830	0	830
180	14	1260	282	978
80	6	610	121	489
120	1	590	20	570
100	9	900	181	719
170	6	820	121	699
110	12	880	241	639
160	7	860	141	719
230	2	760	40	720
70	17	1020	342	678
120	15	1080	302	778
240	7	960	141	819
160	0	700	0	700
90	12	800	241	559
110	16	1130	322	808
220	2	760	40	720
110	6	740	121	619
160	12	980	241	739
80	15	800	302	498

It is evident from Figure 28 that size has a more marked effect upon income than appeared in Figure 25, where the effect of cows was mixed in also. As was pointed out earlier, the fact that cows and acres were correlated meant that the effects of differences in cows were mixed in with the effects of differences in acres. Now that the effect of cows has been at least roughly removed, the change in incomes with changes in acres can be more accurately determined.

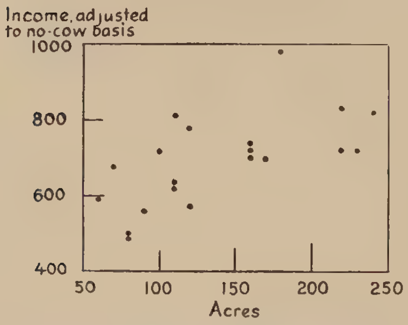

Fig. 28.—Relation of income, adjusted for number of cows, to number of acres.

Fitting straight-lines to the relations shown in Figures 25 and 28, to determine the average change in income with changes in acres, regression equations are obtained, as follows:

Income = \$868.74 + (number of acres) \$0.0234

Income, effect of cows removed,
= \$508.51 + (number of acres) \$1.33

It is evident that the determination of the effect of acres upon income without making some allowance for the effect of the correlated variable, number of cows, in this case would have seriously underestimated the effect of acres upon income. Such a determination would have shown only \$0.02 increase in income for each acre increase in size, whereas the later determination shows \$1.33 increase in income for each acre increase in size.

The relation now shown between income and acres illustrates the extent to which one variable may really influence a second, even though its influence is concealed by the presence of a third variable. From Figure 25, which indicates that there is practically no correlation between acres and income, one might conclude that differences in income were not at all associated with differences in acreage: yet when the variation in income associated with cows is removed, even by the rough method shown, then a very definite relation of income to size is found. For that reason one cannot conclude that because two variables have no correlation, they are not associated with each other; the lack of correlation may be due to the compensating influence of one or more other variables, concealing the hidden relation.

Eliminating the Approximate Influence of Both Variables.—We now have two equations, one showing the effect of cows upon income, and the other the effect of acres:

(A) Income = \$694 + (number of cows) \$20.11

(B) Income, effect of cows removed,
= \$508.51 + (number of acres) \$1.33

These two equations can be combined into a single equation by taking that part of the first one which shows the increase in income for each cow and adding it to the second one. This gives an equation which includes allowances for both factors, as follows:

(C) Income = \$508.51 + (number of acres) \$1.33
+ (number of cows) \$20.11

The last equation gives a basis for indicating the effect of both acres and cows on income, and for computing the income that might be expected, on the average, with a farm of a given size and with a given number of cows. For example, for a farm of 120 acres and 15 cows, the expected income would work out as follows:

$$\begin{aligned}\text{Income} &= \$508.51 + (120)\ \$1.33 + (15)\ \$20.11\\ &= \$508.51 + \$159.60 + \$301.65 = \$970\end{aligned}$$

If 5 cows more were added, making it 120 acres and 20 cows, the estimated income would be:

$$\begin{aligned}\text{Income} &= \$508.51 + (120)\ \$1.33 + (20)\ \$20.11\\ &= \$1070\end{aligned}$$

Or if 50 acres were added, making 170 acres and 15 cows, the income would be estimated:

$$\text{Income} = \$508.51 + (170)\ \$1.33 + (15)\ \$20.11$$

Equation (C) can be used as illustrated, to work out what income might be expected, on the average, for each of the farms shown in Table 39. The estimated income can then be compared with the actual income, and the difference, if any, determined.

As is illustrated in Table 39, the estimated incomes vary somewhat from the actual. This is just another way of saying that all of the differences in income cannot be accounted for by the effect of differences in acres and in cows, according to the relations summarized in equation (C). This failure of the estimated values to agree exactly with the original values is seen graphically in Figure 28 by the fact that all the dots would not lie exactly along the regression line. Subtracting the estimated values from the actual values gives the residual differences of the actual income above or below the income estimated from the two factors, acres and cows.

Correcting Results by Successive Elimination.—It may now be recalled that even though the incomes were adjusted to eliminate the effect of cows upon income before determining the relation between income and acres, the determination of the relation between income and cows was made without making any allowance for the concurrent effect of acres. Since we now have an approximate measure of the effect of acres determined while eliminating to some extent the effect of cows, we can use that new measure, equation (B), to adjust the in-

TABLE 39

ACTUAL INCOME AND INCOME ESTIMATED FROM NUMBER OF ACRES AND COWS

Acres	Cows	Computation of estimated income:		Estimated income (A) + (C) +$508.51	Actual income	Actual income minus estimated income
		Estimate for acres (acres) $1.33 (A)	Estimate for cows (cows) $20.11 (C)			
60	18	$ 80	$362	$ 950.5	$ 960	$ 9.5
220	0	293	0	801.5	830	28.5
180	14	239	282	1029.5	1260	230.5
80	6	106	121	735.5	610	−125.5
120	1	160	20	688.5	590	− 98.5
100	9	133	181	822.5	900	77.5
170	6	226	121	855.5	820	− 35.5
110	12	146	241	895.5	880	− 15.5
160	7	213	141	862.5	860	− 2.5
230	2	306	40	854.5	760	− 94.5
70	17	93	342	943.5	1020	76.5
120	15	160	302	970.5	1080	109.5
240	7	319	141	968.5	960	− 8.5
160	0	213	0	721.5	700	− 21.5
90	12	120	241	869.5	800	− 69.5
110	16	146	322	976.5	1130	153.5
220	2	293	40	841.5	760	− 81.5
110	6	146	121	775.5	740	− 35.5
160	12	213	241	962.5	980	17.5
80	15	106	302	916.5	800	− 116.5

comes for the effect of the acres, and then get a more accurate measure of the true effect of cows alone upon incomes. This process is shown in Table 40. Here estimates of income are worked out by equation (B) on the basis of acres, showing what the incomes might be expected to average if all of the farms had no cows. The difference between these estimates and the actual incomes may then be considered to be the part due to cows alone, while eliminating the effect of differences in the numbers of acres. On the first farm, for example, equation (B) indicates that with no cows the income for 60 acres should be $588. Subtracting this from the $960 actually received leaves $372 as the income apparently accompanying the 18 cows.

The adjusted incomes may then be plotted on a dot-chart with the number of cows as the other variable, as shown in Figure 29. Com-

TABLE 40

ADJUSTING FARM INCOMES FOR DIFFERENCES IN NUMBER OF ACRES

Size of farm	Size of dairy	Income	Income estimated for acres, with no cows	Income with effects of acreage differences eliminated *
Number of acres	*Number of cows*	*Dollars*	*Number of dollars*	*Number of dollars*
60	18	960	588	372
220	0	830	891	29
180	14	1260	748	512
80	6	610	615	– 5
120	1	590	669	– 79
100	9	900	642	258
170	6	820	735	85
110	12	880	655	225
160	7	860	722	138
230	2	760	815	– 55
70	17	1020	602	418
120	15	1080	669	411
240	7	960	828	132
160	0	700	722	– 22
90	12	800	629	171
110	16	1130	655	475
220	2	760	802	– 42
110	6	740	655	85
160	12	980	722	258
80	15	800	615	185

* Where the actual income is below that expected for a farm of that size with no cows, the deficit is indicated by the minus sign.

paring this figure with Figure 26, where the number of cows was plotted against income without first making any adjustment in the original incomes, it is easy to see how much closer the relation is after making the adjustment. Further, it is evident that cows have a greater effect upon income than was indicated by the earlier comparison. Computing the straight-line relationship for Figure 29 gives the equation:

(D) Income, adjusted to constant acres

$$= -\$68.77 + (\text{number of cows})\ \$27.88$$

By this last computation (equation D), each increase of one cow causes an average increase in income of $27.88, whereas according to the earlier comparison (equation A), each increase of one cow caused an average increase in income of only $20.11. The second value is larger than the first, again showing the necessity of making allowances for the effect of one factor before the true value of the other can be properly measured.

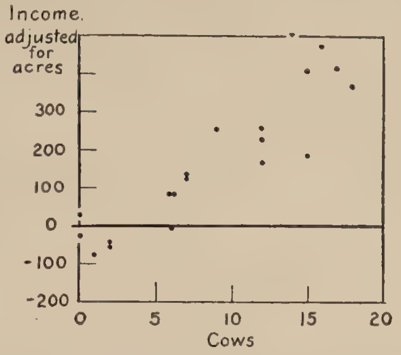

FIG. 29.—Relation of income, adjusted for number of acres, to number of cows.

Now that we have a new measure of the effect of cows, we might go on to adjust incomes for cows by this new measure, and then get a revised value for the effect of acres upon incomes on a no-cow basis, in place of the relation shown in equation (B). This possibility of further correction will be referred to later. But before that we will make some experiments with the new equation (D).

We now have equations for the relation of incomes, adjusted for the other factors, to the remaining factors. These two equations, (B) and (D), are:

(B) Income, effect of cows removed,

$$= \$508.51 + (\text{number of acres})\ \$1.33$$

(D) Income, adjusted to constant acres,

$$= -\$68.77 + (\text{number of cows})\ \$27.88$$

These two equations may be combined to give a revised equation to indicate the effect of both cows and acres upon incomes, equation (E).

$$\text{(E) Income} = \$439.74 + (\text{number of acres})\ \$1.33 + (\text{number of cows})\ \$27.88$$

Equation (E) is exactly the same as the previous equation (C) except that the revised effect of cows is included, and the constant term has also been changed due to changing the allowances for cows.

In exactly the same way that equation (C) could be used to work out the estimated income for any given combination of cows and acres, equation (E) can be also used. Thus for 120 acres and 15 cows, it would give

$$\text{Estimated income} = \$439.7 + (120)\ \$1.33 + (15)\ \$27.88$$
$$= \$439.7 + \$159.6 + \$418.2 = \$1{,}018$$

The result, \$1,018, is \$48 higher than the \$970 worked out by equation (C). This higher estimate is due to the fact that equation (E) makes a larger allowance for the effect of each cow, and 15 is more than the average number of cows. If less than the average number of cows were used, equation (E) would give a lower estimate than equation (C).

Working out the estimated incomes for each of the original observations according to equation (E), the results come out as shown in Table 41.

TABLE 41

ACTUAL INCOME AND INCOME ESTIMATED FROM NUMBER OF ACRES AND NUMBER OF COWS, REVISED RELATIONS

Acres	Cows	Computation of estimated income		Estimated income, (A) + (B) + \$439.7	Actual income	Actual income minus estimated income
		Estimate for acres (acres) \$1.33 (A)	Estimate for cows (cows) \$27.88 (B)			
60	18	\$ 80	\$502	\$1021.7	\$ 960	−\$ 61.7
220	0	293	0	732.7	830	97.3
180	14	239	390	1068.7	1260	191.3
80	6	106	167	712.7	610	−102.7
120	1	160	28	627.7	590	− 37.7
100	9	133	251	823.7	900	76.3
170	6	226	167	832.7	820	− 12.7
110	12	146	335	920.7	880	− 40.7
160	7	213	195	847.7	860	12.3
230	2	306	56	801.7	760	− 41.7
70	17	93	474	1006.7	1020	13.3
120	15	160	418	1017.7	1080	62.3
240	7	319	195	953.7	960	6.3
160	0	213	0	652.7	700	47.3
90	12	120	335	894.7	800	− 94.7
110	16	146	446	1031.7	1130	98.3
220	2	293	56	788.7	760	− 28.7
110	6	146	167	752.7	740	− 12.7
160	12	213	335	987.7	980	− 7.7
80	15	106	418	963.7	800	−163.7

Comparing the residuals, or differences between the actual and estimated income, obtained by means of this new equation with those obtained using the equation in its first form (shown in Table 39), we see that in more than half of the cases they are smaller with the revised form. A more definite comparison can be made by computing the standard deviation of the residuals in each case. The standard deviation of the residuals shown in Table 39, using equation (C), is $90.29, whereas the standard deviation of the residuals shown in Table 41, using equation (E) is but $78.70. It is apparent from this that the revised equation, determined after the effects of the other variables had been eliminated, gives more accurate estimates of income than does the original equation, in which the effects of the other variables had not been so fully eliminated.

It was suggested previously that the last corrected values for the relation of cows to income gave a new basis for correcting income so as to measure more accurately the relation of acres to income. This in turn would give a new basis for measuring the effect of cows, and so on, until a final stable value had been reached. So long as a new correction would result in a further change in the computed effect of either variable, the new values would give a better basis for estimating income than did the previous values. Only when the point was reached where no further change need be made in the effect of either variable could it be said that the relation of each variable to income had been quite correctly measured while allowing for the influence of the other factor, and that might involve a large number of successive corrections.

This method of allowing for the effect of other factors so as to determine the true relation of each one to the dependent factor (as income, in this case), by first correcting for one, and then for another, is known as the method of successive elimination. This method can be used where there are three or more independent factors related to (or accompanying variations in) a dependent (or resultant) factor just as it was used here for two factors, except that then the dependent needs to be corrected in turn to eliminate the effects of all of the other independent factors except the particular one whose effect is being measured. But while it is possible to measure the relations by this method, it would be a very slow and laborious process; and a shorter mathematical method which gives the same result by more direct processes is available instead. This method, known as the method of multiple correlation, is presented in detail in Chapter 12.

Summary.—This chapter has shown that when two related factors both affect a third factor it is difficult to measure the effect of either factor upon the third without the result being affected by both causal factors. Allowing for this duplication by eliminating the effects of each factor in turn (successive elimination) can gradually determine the true effect of each, but the method is long and laborious.

CHAPTER 11

DETERMINING THE WAY ONE VARIABLE CHANGES WHEN TWO OR MORE OTHER VARIABLES CHANGE

(2) By cross-classification and averages

We have previously seen (Chapter 4) how the relation between two variables can be studied by means of averages. An extension of the same method can be used for problems where two or more variables affect a third variable, such as that discussed in the last chapter.

Analysis by averages where there are two independent variables involves classifying the records first by one variable, then breaking each of the resulting groups into several smaller groups according to the values of the second variable. If a third independent variable were to be considered, these groups would be broken up into still smaller groups, according to the values of the third variable. Then the values of the dependent variable, as well as each of the independent variables, would be averaged for each sub-group. This process is known as sub-classification or cross-classification.

Cross-classification for Three Variables.—In the problem presented in the last chapter, there were two independent variables—number of cows and number of acres. The records would therefore need to be classified into groups both according to the number of cows and the number of acres on each farm. Since there is such a small number of records the groups should not be made too small. Let us take three groups for cows: less than 6, 6 to 11, and 12 and over; and four groups for the size of farm; from 50 to 99 acres, from 100 to 149, from 150 to 199, and 200 acres and over. This will give us twelve possible groups in all. The records may be classified into these twelve groups and totals and averages computed for each, as shown in detail in Table 42.

It is apparent that none of these groups has a sufficient number of farms represented to make the averages particularly significant; yet even at that a certain regularity in the averages can be observed. In each column the average income increases as the size of farm increases, though there is but little difference in the average number of cows

TABLE 42

CROSS-CLASSIFICATION OF INCOME ACCORDING TO SIZE OF FARM AND SIZE OF DAIRY HERD

Size of farm	Size of dairy herd								
	Under 6 cows			6 to 11 cows			12 cows and over		
	Acres	Cows	Income	Acres	Cows	Income	Acres	Cows	Income
	Number	*Number*	*Dollars*	*Number*	*Number*	*Dollars*	*Number*	*Number*	*Dollars*
50 to 99 acres.	......	......	......	80	6	610	60	18	960
	......	......	......	......	......	......	70	17	1020
	......	......	......	......	......	......	90	12	800
	......	......	......	......	......	......	80	15	800
Total........	......	......	......	......	......	......	300	62	3580
Average......	......	......	......	80	6	610	75	15.5	895
100 to 149 acres	120	1	590	100	9	900	110	12	880
	......	......	......	110	6	740	120	15	1080
	......	......	......	......	......	......	110	16	1130
Total........	......	......	......	210	15	1640	340	43	3090
Average......	120	1	590	105	7.5	820	113	14.3	1030
150 to 199 acres	160	0	700	170	6	820	180	14	1260
	......	......	......	160	7	860	160	12	980
Total........	......	......	......	330	13	1680	340	26	2240
Average......	160	0	700	165	6.5	840	170	13	1120
200 acres and over.....	220	0	830	240	7	960			
	230	2	760						
	220	2	760						
Total........	670	4	2350						
Average......	223	1.3	783	240	7	960			

from group to group; similarly across each line of averages the income increases as the number of cows increases, though there is but little difference in the average size of farm from group to group. These relations may be more clearly seen in Figures 30 and 31, where the average incomes from Table 42 are charted, first for differences in the number of cows with farms of similar sizes, and then for differences in the number of acres, with farms of similar numbers of cows.

Both figures show the tendency for income to increase with an increase in the independent variable, when the effect of the other variable is held fairly constant by the grouping process. In Figure 30 the lines show about the same general slope for each of the four groups, though there are some irregularities. Figure 31 similarly shows about the same general change in income with a given change in the size of the farm, no matter what is the number of cows; but here the irregularities from group to group are even more striking.

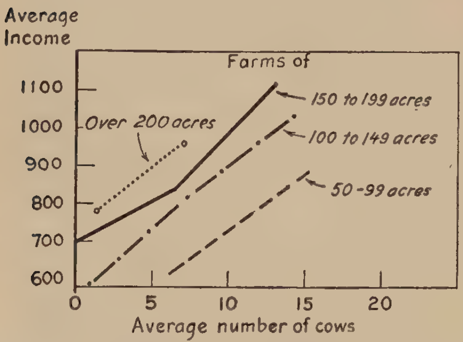

Fig. 30.—Difference in average income with difference in number of cows, for farms grouped by size of farm.

In Chapter 4 it was shown that such irregularities from group to group might readily be due to random errors of sampling. In the present case, the number of items in each group is so small that it would be hardly worth while to compute the standard error for each average. Even if there were many more cases in each group than are available here, differences as large as those shown might be due simply to random differences in sampling and therefore have no real meaning as indicating differences prevailing in the universe from which the sample was selected.

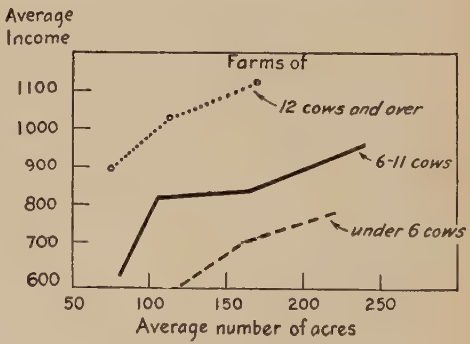

Fig. 31.—Difference in average income with difference in number of acres, for farms grouped by numbers of cows.

Although the averages obtained by the process of subsorting may be considered to show the general effect of changes in one variable, such as cows, upon income, with the effect of the other variable, such as acres, removed, they cannot be considered to show the specific effect of specific differences. For example, much more evidence would be needed to prove that, between 75 and 100 acres, a change of 1 acre has much greater effect upon income on farms with 6 to 11 cows than on farms with 12 cows or more, even though the lines in Figure 31 would appear to indicate this. All that is really proved is that on farms of both sizes there is a tendency for income to increase with an increase in the number of acres.

The averages obtained by the process shown in Table 42 may be summarized for publication in a form similar to Table 43. The number of cases represented in each average is included to prevent the reader from placing an undue amount of confidence in an average based on a small number of observations.

TABLE 43

DIFFERENCE IN AVERAGE INCOME FOR FARMS OF DIFFERENT SIZES AND WITH DIFFERENT SIZES OF DAIRY HERD

Size of farm	Under 6 cows in herd		6 to 11 cows in herd		12 cows or over in herd	
	Size of group	Average income	Size of group	Average income	Size of group	Average income
	Number of farms	*Dollars*	*Number of farms*	*Dollars*	*Number of farms*	*Dollars*
50 to 99 acres...	....	....	1	610	4	895
100 to 149 acres...	1	590	2	820	3	1030
150 to 199 acres...	1	700	2	840	2	1120
200 to 249 acres...	3	783	1	960	....	....

The very small number of cases included in each of the groups is strikingly brought out in Table 43. Even if there were five times as many farms to deal with—100 in all—if they were distributed in the same manner, the largest group would only have 20 cases, and all the rest would have 15 or less, which, under ordinary conditions, would be hardly enough for really significant averages.

Limitation of Cross-classification for Many Variables.—This small problem illustrates one fundamental difficulty with the method of sub-classification and averaging—the large number of cases required for conclusive results. Though there are only two independent variables involved, and the records are classified into only three groups one way and four the other, apparently 100 cases or more would be required for really significant results. If it had been desired to sub-

TABLE 44

Form for Showing Differences in Average Income for Farms Classified by Acres, Men Employed, Cows and Hogs

Area and number of hogs	Under 6 cows					
	1 man		2 men		3 men	
	Size *	Average income	Size *	Average income	Size *	Average income
Farms of 50 to 99 acres:						
Under 20 hogs........						
20–39 hogs...........						
40 hogs and over......						
Farms of 100 to 149 acres:						
Under 20 hogs........						
20–39 hogs...........						
40 hogs and over......						
	6 to 11 cows					
Farms of 50 to 99 acres:						
Under 20 hogs........						
20–39 hogs...........						
40 hogs and over......						
Farms of 100 to 149 acres:						
Under 20 hogs........						
20–39 hogs...........						
40 hogs and over......						
	Over 12 cows					
Farms of 50 to 99 acres:						
Under 20 hogs........						
20–39 hogs...........						
40 hogs and over......						
Farms of 100 to 149 acres:						
Under 20 hogs........						
20–39 hogs...........						
40 hogs and over......						

Etc.

* Number of reports in group.

classify the records according to two more additional variables—say number of men employed and number of hogs kept—that would have greatly increased the number of records necessary. If each of the groups already shown had been further divided into 1-man, 2-men, and 3-or-more-men farms, and each of these sub-groups had been further divided into farms with less than 20 hogs, 20 to 39 hogs, and 40 or more hogs, that would have increased the number of possible groups from 12 to 108. Where over 100 records would have been needed to give at all reliable results in the first case, probably a thousand or more records would be needed with this further classification. Although such large numbers of records are available in some types of work, as in census tabulations, they are rarely obtainable in most economic or social science studies, and for that reason treatment of a large number of variables by the method of detailed sub-classification has but limited application in this field.

The way in which a four-fold classification, such as that described in the preceding paragraph, might be presented is indicated by the form in Table 44, even though it would only occasionally be used.

In addition to the large number of cases required to obtain reliable results, the method of sub-classification and averaging has one further shortcoming; it provides no measure of how *important* the relation shown is as a cause of variation in the factor being studied, nor of how closely that factor may be estimated from the others on the basis of the relations shown. Thus Table 43 shows that, on the average, certain differences in the number of cows and in the number of acres were accompanied by certain differences in the average income. By itself, however, it did not give any indication of how closely the income could be estimated if the number of acres or the number of cows were known; nor did it indicate the proportion of the variance in income which can be explained by concurrent differences in size of farm and size of dairy. For these reasons, as well as because of the large number of cases necessary to obtain reliable conclusions, the method of sub-classification and averaging does not determine the relationships where many variables are involved so satisfactorily as do other methods, which will be considered in subsequent chapters.

Summary.—The relation of one variable to several others may be approximately determined by detailed cross-classification. Very large numbers of records are required to make the averages accurate, however, since the number of groups increases rapidly with additional variables. Further, the averages by themselves give no indication of the closeness of correlation.

CHAPTER 12

DETERMINING THE WAY ONE VARIABLE CHANGES WHEN TWO OR MORE OTHER VARIABLES CHANGE

(3) By using a linear regression equation

In Chapter 10 it was shown that an equation could be arrived at to express the average relation between income, acres, and cows, as follows: Equation (E)

$$\text{Income} = 439.74 + 1.33 \text{ (number of acres)} + 27.88 \text{ (number of cows)}$$

If we designated the three series of variable quantities, income, acres, and cows, by the symbol X with different subscripts, using X_1 to represent dollars of income, X_2 to represent number of acres, and X_3 to represent the number of cows, we can rewrite the equation in the form

$$X_1 = 439.74 + 1.33X_2 + 27.88X_3$$

If now we use the symbol a to represent the constant quantity 439.74; b_2 to represent 1.33, the amount which X_1 increases for each increase of one unit in X_2 (one acre); and b_3 to represent 27.88, the amount which X_1 increases for each increase of one unit in X_3 (one cow); the equation appears

$$X_1 = a + b_2X_2 + b_3X_3 \qquad (30)$$

Comparing this equation with the regression equation for the straight-line relation between two variables

$$Y = a + bX$$

we see that (except for the difference in the symbols used to represent the different variables) the two equations are just alike, except that the expression for an additional variable has been added to the simpler equation. In equation (30), X_1, the variable which is being estimated, is termed the *dependent* variable, since its estimated value depends upon those of the other variable or variables; and X_2 and X_3 are termed *independent* variables, since their values are taken just as observed, independent of any of the conditions of the problem. Since there is more than one independent variable concerned, the equation is said to be a multiple estimating equation, or a multiple regression equation.

Chapter 10 showed that the values of the constants a, b_2, and b_3, which in the particular problem considered indicate what the average income would be for any given size of farm and size of dairy, could be worked out by a cut-and-try method which gradually approached nearer and nearer to the right values. It is evident, however, that for any particular criterion of "rightness" only one set of values for these constants can be exactly right. If the criterion of "rightness" is taken as that which will make the standard deviation of the residuals, when income is estimated from the other two variables, as small as possible, then the values of a, b_2, and b_3 which will give this result can be determined once and for all by a direct mathematical process. Determining these values so as to give the "best" equation for estimating X_1 on the basis of linear relations to X_2 and X_3, is the first step in the method of *linear multiple correlation.*

Determining a Regression Equation for Two Independent Variables.—The best values for a, b_2, and b_3 in the multiple regression equation (30), can be worked out by an extension of the same process used in working out the values for the estimating equation when only one independent variable was considered. Just as before, the value of the b constants will be determined first, equation (31), and then the a values will be worked out from them:[1]

$$\left.\begin{aligned} \Sigma(x_2^2)b_2 + \Sigma(x_2x_3)b_3 &= \Sigma(x_1x_2) \\ \Sigma(x_2x_3)b_2 + \Sigma(x_3^2)b_3 &= \Sigma(x_1x_3) \end{aligned}\right\} \qquad (31)$$

$$a = M_1 - b_2M_2 - b_3M_3. \qquad (32)$$

Here, just as in Chapter 5, the symbol M represents the mean value of each variable, and the subscript indicates the particular variable.

Similarly, the symbols $\Sigma(x_2x_3)$, $\Sigma(x_1x_2)$, and $\Sigma(x_1x_3)$ represent the sums of the products of the variables, corrected to adjust them to deviations from the mean; that is, $\Sigma(x_1x_2) = \Sigma[(X_1 - M_1)(X_2 - M_2)]$. Likewise the symbols $\Sigma(x_2^2)$, etc., represent the sums of the squares of the variables, also adjusted to deviations from the mean.

Using the two basic formulas

$$\Sigma(x_1x_2) = \Sigma(X_1X_2) - nM_1M_2 \qquad (11)$$

and

$$\Sigma(x_2^2) = \Sigma(X_2^2) - n(M_2^2)$$

the other values shown in equation (31) may be worked out as follows:

$$\Sigma(x_1x_3) = \Sigma(X_1X_3) - nM_1M_3$$

$$\Sigma(x_2x_3) = \Sigma(X_2X_3) - nM_2M_3$$

$$\Sigma(x_3^2) = \Sigma(X_3^2) - n(M_3^2)$$

---

[1] See Technical Appendix, Note 6, for the derivations of these equations.

Computing the extensions.—Inspection of these equations shows that there are eight arithmetic values which must be computed from the original data to work out the values to substitute in equations (31) and (32). These are $\Sigma(X_1)$, $\Sigma(X_2)$, $\Sigma(X_3)$, $\Sigma(X_2^2)$, $\Sigma(X_3^2)$, $\Sigma(X_1X_2)$, $\Sigma(X_1X_3)$, and $\Sigma(X_2X_3)$. The actual work of computing these values for the farm income data originally presented in Table 35, is shown in Table 45. [The value $\Sigma(X_1^2)$ is not needed in solving equations (31) or (32), but as it will be needed later, it is also worked out here, for convenience in calculation.]

TABLE 45

Computation of Values to Determine Multiple Regression Equation to Estimate One Variable from Two Others

	Number of acres * X_2	Number of cows X_3	Number of dollars income * X_1	X_2^2	X_2X_3	X_1X_2	X_3^2	X_1X_3	X_1^2
	6	18	96	36	108	576	324	1,728	9,216
	22	0	83	484	0	1,826	0	0	6,889
	18	14	126	324	252	2,268	196	1,764	15,876
	8	6	61	64	48	488	36	366	3,721
	12	1	59	144	12	708	1	59	3,481
	10	9	90	100	90	900	81	810	8,100
	17	6	82	289	102	1,394	36	492	6,724
	11	12	88	121	132	968	144	1,056	7,744
	16	7	86	256	112	1,376	49	602	7,396
	23	2	76	529	46	1,748	4	152	5,776
	7	17	102	49	119	714	289	1,734	10,404
	12	15	108	144	180	1,296	225	1,620	11,664
	24	7	96	576	168	2,304	49	672	9,216
	16	0	70	256	0	1,120	0	0	4,900
	9	12	80	81	108	720	144	960	6,400
	11	16	113	121	176	1,243	256	1,808	12,769
	22	2	76	484	44	1,672	4	152	5,776
	11	6	74	121	66	814	36	444	5,476
	16	12	98	256	192	1,568	144	1,176	9,604
	8	15	80	64	120	640	225	1,200	6,400
Sums..	279	177	1,744	4,499	2,075	24,343	2,243	16,795	157,532
Means.	13.95	8.85	87.2						
Correction item..........				3,892.05	2,469.15	24,328.80	1,566.45	15,434.40	152,076.80
Corrected sums..........				606.95	− 394.15	14.20	676.55	1,360.60	5,455.20

* In these computations, X_2 and X_1 have been divided by 10. (See Technical Appendix, Note 3.)

After multiplying through all the extensions shown in this table, and adding each of the columns, the next step is to compute the values

5
85
4575
20
169.150
075
394.15

M_2, M_3, and M_1, by dividing the sums of each of the first three columns by the number of cases. The correction values for each of the products is then computed and entered below the value from which it is to be subtracted. Thus the value below the sum of the fourth column, $\Sigma(X_2^2)$, is its correction factor, $n(M_2^2)$. This is equal to $20(13.95)^2$, or 3892.05, which is the value entered. Similarly, the value below the sums of the fifth column, $\Sigma(X_2X_3)$, is its correction factor $n(M_2M_3)$, or 20(8.85)(13.95), which equals 2469.15. All the other correction factors are similarly worked out and entered. Then subtracting each correction factor from the value above it gives the values all ready for equations (31). Thus the value at the foot of column 4 is the value for $\Sigma(x_2^2)$; and so on. Substituting these values in the appropriate spaces of equations (31), they become

$$\left.\begin{aligned}\Sigma(x_2^2)b_2+\Sigma x_2x_3b_3&=\Sigma x_1x_2\\ \Sigma x_2x_3b_2+\Sigma(x_3^2)b_3&=\Sigma x_1x_3\end{aligned}\right\}=\left\{\begin{aligned}606.95\,b_2-394.15\,b_3&=14.20 \quad \text{(I)}\\ -394.15\,b_2+676.55\,b_3&=1360.60 \quad \text{(II)}\end{aligned}\right.$$

Solving the equations.—The next step is to solve the two algebraic equations simultaneously to determine the values for b_2 and b_3.

The simplest way to carry this through is by the Doolittle method. The first equation is divided through by the coefficient of b_2, with the sign changed, giving the first derived equation (I′):

$$\begin{array}{lrcl}\text{(I)} & 606.95\,b_2 - 394.15\,b_3 &=& 14.20\\ \text{(I}'\text{)} & -\,b_2 + 0.64939\,b_3 &=& -\,0.02340\end{array}$$

Then equation (II) is entered, and under it is written equation (I′) multiplied by the coefficient of b_3 in equation (I) (−394.15). The sum of these two equations is then taken, eliminating the values in b_2:

$$\begin{array}{rlrcl} & \text{(II)} & -394.15\,b_2 + 676.55\,b_3 &=& 1360.60\\ (-394.15) & \text{(I}'\text{)} & +394.15\,b_2 - 255.96\,b_3 &=& 9.22\\ & (\Sigma\text{II}) & 420.59\,b_3 &=& 1369.82\\ & \text{(II}'\text{)} & b_3 &=& 3.25690\end{array}$$

As indicated above, this step gives the value of b_3. This is then substituted in equation (I′) and the value of b_2 determined

$$-\,b_2 + 0.64939(3.25690) = -\,0.02340$$
$$b_2 = 0.02340 + 2.11500 = 2.13840$$

The values of b_2 and b_3 being thus obtained, the next step is to substitute them, together with the other values required, in equation (32) to work out the value for a:

$$a = M_1 - b_2M_2 - b_3M_3$$

$$\begin{aligned} a &= 8.72 - (2.1384)(13.95) - (3.2569)(8.85) \\ &= 87.2 - 29.83 - 28.82 = 28.55 \end{aligned}$$

Estimating X_1 *from* X_2 *and* X_3.—Having computed the values for a, b_2, and b_3, we can now write out our regression equation, (30), with the *best* values, as determined by the mathematical calculation:

$$\left(\frac{X_1}{10}\right) = 28.55 + 2.138\left(\frac{X_2}{10}\right) + 3.2569X_3$$

$$X_1 = 285.5 + 2.138X_2 + 32.569X_3$$

Comparing this equation with the last one obtained in Chapter 10, we see that the mathematical determination has changed the \$1.33 allowed for the effect of each acre (b_2) to \$2.14, and increased the \$27.88 allowed for the effect of each cow (b_3) to \$32.57. Just what effect this has on the accuracy of the equation as a basis for estimating income from cows and acres may be judged by working out an estimated income for each of the 20 cases according to these last results, and then comparing the estimated values with the original values, just as was done before with the equations worked out by the approximation method. The necessary computation is shown in Table 46.

The operations that have been performed in this table may be mathematically stated as follows:

First, an estimated value of income, X_1, has been worked out by substituting in equation (30) the values for X_2 and X_3 given by each successive observation. Using the symbol X'_1 to represent this estimated value of X_1 it may be defined

$$X'_1 = a + b_2X_2 + b_3X_3 \tag{33}$$

Each estimated income has next been subtracted from the corresponding actual income. Using the symbol z to represent the *residual*, the amount by which the actual value exceeds or falls below the estimated value, it may be defined

$$z = X_1 - X'_1 \tag{34}$$

The residual z has exactly the same meaning when the estimated values of the dependent variable are based upon two or more variables, using multiple correlation, as it had previously when the estimate was based on a single variable, with simple correlation.

TABLE 46

Actual Income and Income Estimated from Number of Acres and Cows, on Basis of Mathematically Determined Relations

Acres, X_2	Cows, X_3	Computation of estimated incomes			Estimated income X'_1	Actual income X_1	Actual minus estimated income, $X_1 - X'_1$ z
		Estimated for acres, b_2X_2	Estimated for cows, b_3X_3	Constant, a			
60	18	128	586	286	1000	960	−40
220	0	470	....	286	756	830	74
180	14	385	456	286	1127	1260	133
80	6	171	195	286	652	610	−42
120	1	257	33	286	576	590	14
100	9	214	293	286	793	900	107
170	6	363	195	286	844	820	−24
110	12	235	391	286	912	880	−32
160	7	342	228	286	856	860	4
230	2	492	65	286	843	760	−83
70	17	150	554	286	990	1020	30
120	15	257	489	286	1032	1080	48
240	7	513	228	286	1027	960	−67
160	0	342	....	286	628	700	72
90	12	192	391	286	869	800	−69
110	16	235	521	286	1042	1130	88
220	2	470	65	286	821	760	−61
110	6	235	195	286	716	740	24
160	12	342	391	286	1019	980	−39
80	15	171	489	286	946	800	−146

The accuracy of the last estimating equation, derived by an exact mathematical process, can now be compared with the accuracy of previous equations, obtained by a cut-and-try process. Computing the standard deviation of the residuals shown in this last table and comparing it with the standard deviations of the residuals worked out in Tables 39 and 41 of Chapter 10, the comparison is as follows:

Standard deviations of residuals using various straight-line equations:

First approximation equation, $\sigma_z = 90.29$

Second approximation equation, $\sigma_z = 78.70$

Mathematically determined equation, $\sigma_z = 70.48$

The equation determined mathematically gives a much closer estimate of the actual incomes from which it was derived than do either of the two previous equations. This will always hold true. The mathematically determined equation gives once and for all the estimates of X_1 which will make σ_z the smallest that can be obtained, assuming linear relations. The best that could be done by the approximation method would be to obtain the same conclusions as would be obtained by the other method. The successive steps in Chapter 10 have shown how difficult it is to do this when the several independent variables are correlated with each other, and so tend to vary with one another. The mathematical method for determining the estimating equation, as illustrated in this Chapter (or some alternative form of computation involving the same principle) has therefore been practically universally adopted as the standard way of determining the precise way in which one variable is related to, or may be estimated from, two or more variables related among themselves, if only straight-line relations are to be assumed.

When the constants of the estimating equation are determined by the exact mathematical process, the equation is then called a "multiple regression equation," and the constants b_2 and b_3, which show, in this case, the average increase in income (X_1) for unit increases in acres (X_2), and cows (X_3), are termed "net regression coefficients." The constant b_2 is termed "the net regression of X_1 on X_2, holding X_3 constant," and b_3 is termed "the net regression of X_1 on X_3, holding X_2 constant." All that that means for b_2, for example, is "the average change observed in X_1, with unit changes in X_2, determined while simultaneously eliminating from X_1 any variation accompanying (hence temporarily assumed due to) changes in X_3."

In order that the mathematical notation for the net regression coefficients may show quite clearly which independent variables were held constant when a particular coefficient was determined, the subscripts under the b are sometimes more elaborate, showing first the dependent variable, then the independent variable whose effect is stated, then a period followed by the independent variables which were held constant in the process. Thus the b_2 we have been using would be written $b_{12.3}$. The whole regression equation would appear

$$X_1 = a_{1.23} + b_{12.3}X_2 + b_{13.2}X_3 \tag{35}$$

This notation serves to distinguish these net regression coefficients from those which would be obtained if additional independent variables were included. Thus if a third independent variable, say X_4, were also considered, the equation would read

$$X_1 = a_{1.234} + b_{12.34}X_2 + b_{13.24}X_3 + b_{14.23}X_4 \tag{36}$$

For still another variable it would be

$$X_1 = a_{1.2345} + b_{12.345}X_2 + b_{13.245}X_3 + b_{14.235}X_4 + b_{15.234}X_5 \qquad (37)$$

The notation for a is changed as well as for each of the b's; $a_{1.234}$ will probably be a different value from $a_{1.23}$, just as $b_{12\cdot34}$ is likely to be somewhat different from $b_{12.3}$. This is to be expected; if some other factor, such as the number of men working on each farm, were taken into account as well as the number of acres and the number of cows, the average increase in income per additional acre, with both the number of cows and the number of men held constant, might be quite different from what it would be with only the number of cows held constant. In the last case, any increase in income due to there being more men at work on the larger number of acres would be ascribed to the acres and not to the men, whereas in the former this element would be removed from the increase attributed to the acres.

Determining a Regression Equation for Three Independent Variables.—Solely to illustrate the method, we may take the number of men on each of these 20 farms as given in Table 47, and work out an estimating equation considering men as well as acres and cows. (In actual practice, 20 observations are usually too few to determine, with any degree of reliability, the net relation of one variable to 3 independent variables. This problem is used here solely to carry through the same illustration.)

Designating the number of men as X_4, the unknown constants to be determined are those given in equation (36); $a_{1.234}$, $b_{12\cdot34}$, $b_{13.24}$, and $b_{14.23}$. They can be obtained by the solution of the following set of equations.

$$\left.\begin{aligned}
\Sigma(x_2^2)b_{12.34} + \Sigma(x_2x_3)b_{13.24} + \Sigma(x_2x_4)b_{14.23} &= \Sigma(x_1x_2)\\
\Sigma(x_2x_3)b_{12.34} + \Sigma(x_3^2)b_{13.24} + \Sigma(x_3x_4)b_{14.23} &= \Sigma(x_1x_3)\\
\Sigma(x_2x_4)b_{12.34} + \Sigma(x_3x_4)b_{13.24} + \Sigma(x_4^2)b_{14.23} &= \Sigma(x_1x_4)
\end{aligned}\right\} \qquad (38)$$

$$a_{1.234} = M_1 - b_{12.34}M_2 - b_{13.24}M_3 - b_{14.23}M_4 \qquad (39)$$

Computing the extensions.—All except 4 of the arithmetic values for equation (38) which need to be calculated from the original data have been worked out previously. Only the values which involve X_4, and its mean, are additional. The new values needed are therefore M_4, $\Sigma(x_1x_4)$, $\Sigma(x_2x_4)$, $\Sigma(x_3x_4)$, and $\Sigma(x_4^2)$. The computation of these values is shown in Table 47.

All the calculations, including correcting for the means at the end, are carried out just as with Table 45. The figures at the foot of each column provide the remaining values necessary to write out equations

TABLE 47

Computation of Additional Values to Determine Multiple Regression Equation, Adding a Third Independent Factor

Item number	Number of acres, X_2*	Number of cows, X_3	Number of men, X_4	Number dollars income, X_1*	X_2X_4	X_3X_4	X_1X_4	X_4^2
1	6	18	2	96	12	36	192	4
2	22	0	3	83	66	0	249	9
3	18	14	4	126	72	56	504	16
4	8	6	1	61	8	6	61	1
5	12	1	1	59	12	1	59	1
6	10	9	1	90	10	9	90	1
7	17	6	3	82	51	18	246	9
8	11	12	2	88	22	24	176	4
9	16	7	2	86	32	14	172	4
10	23	2	3	76	69	6	228	9
11	7	17	2	102	14	34	204	4
12	12	15	3	108	36	45	324	9
13	24	7	4	96	96	28	384	16
14	16	0	2	70	32	0	140	4
15	9	12	1	80	9	12	80	1
16	11	16	3	113	33	48	339	9
17	22	2	2	76	44	4	152	4
18	11	6	1	74	11	6	74	1
19	16	12	2	98	32	24	196	4
20	8	15	2	80	16	30	160	4
Sums.....	279	177	44	1744	677	401	4030	114.00
Means....	13.95	8.85	2.2	87.2				
Correction items					613.80	389.40	3836.80	96.80
Corrected sums					63.20	11.60	193.20	17.20

* Coded by dividing by 10.

(38) in full. For convenience in writing these equations, we will again use the abridged notation of b_2 for $b_{12.34}$, b_3 for $b_{13.24}$, etc., remembering, however, that b_2 here is a different constant from b_2 previously.

$$\left.\begin{array}{ll}
\text{(I)} & \Sigma(x_2^2)b_2+\Sigma(x_2x_3)b_3 \\
 & \qquad +\Sigma(x_2x_4)b_4=\Sigma(x_1x_2) \\
\text{(II)} & \Sigma(x_2x_3)b_2+\Sigma(x_3^2)b_3 \\
 & \qquad +\Sigma(x_3x_4)b_4=\Sigma(x_1x_3) \\
\text{(III)} & \Sigma(x_2x_4)b_2+\Sigma(x_3x_4)b_3 \\
 & \qquad +\Sigma(x_4^2)b_4=\Sigma(x_1x_4)
\end{array}\right\} = \left\{\begin{array}{l}
606.95b_2-394.15b_3 \\
\qquad +63.20b_4=14.20 \\
-394.15b_2+676.55b_3 \\
\qquad +11.60b_4=1360.60 \\
63.20b_2+11.60b_3 \\
\qquad +17.20b_4=193.20
\end{array}\right.$$

Solving the equations.—The three equations are now to be solved simultaneously to determine the values for b_2, b_3, and b_4. This can be done by the usual algebraic processes, but the peculiar symmetrical character of the equations, which the attentive reader has probably already noticed, makes it possible to use a much shorter method. Since the saving in clerical labor by the use of this method is quite significant, it will be shown in full.

The first step is to set down the first equation (I) and divide it through by the coefficient of the first term, Σx_2^2, *with the sign changed*, or -606.95 in this case. The resulting derived equation (I′) is set down just below it:

$$\begin{array}{lrcl}
\text{(I)} & 606.95b_2 - 394.15b_3 + 63.20b_4 & = & 14.20 \\
\text{(I}'\text{)} & -b_2 + 0.64939b_3 - 0.10413b_4 & = & -0.02340
\end{array}$$

The next step is to set down the second equation (II). The first equation (I) is then multiplied by the coefficient of the *second term* in the derived equation (I′), which is $+0.64939$ in this case, and the products set down just below equation (II). These two equations are added, giving the sum equation (Σ_2) which cancels out the first term, as shown below. The sum equation is then divided by the coefficient of its first term, with the sign changed, giving the second derived equation (II′). The second portion of the work now appears as follows:

$$\begin{array}{rrcr}
\text{(II)} & -394.15b_2 + 676.55b_3 + 11.60b_4 & = & 1360.60 \\
(0.64939)\text{(I)} & 394.15b_2 - 255.96b_3 + 41.04b_4 & = & 9.22 \\
\hline
(\Sigma_2) & 420.59b_3 + 52.64b_4 & = & 1369.82 \\
\text{(II}'\text{)} & -b_3 - 0.12516b_4 & = & 3.25690
\end{array}$$

The final step in the process of elimination is to write down equation (III), multiply the first equation (I) by the coefficient of the *third* term of the first derived equation (I′), which is -0.10413 in this case, and set the products down below equation (III); multiply the sum equation (Σ_2) by the corresponding coefficient (the second term) from the second derived equation (II′), -0.12516; and set these products down below the previous equation. Equation (III) and the two new equations are then added, giving an equation (Σ_3), from which values in both b_2 and b_3 have been eliminated. This equation is then divided by the coefficient of its first term, with the sign changed, -4.03 in this case, and the resulting new derived equation entered as equation (III′). (A method of checking each step in these computations is shown in Appendix I, Methods of Computation, page 362.) All of the computations to this point are as follows:

$$
\begin{array}{r|r|r|r|cr}
\text{(I)} & 606.95b_2 & -\,394.15b_3 & +\,63.20b_4 & = & 14.20 \\
\text{(I}'\text{)} & -\,b_2 & +\,0.64939b_3 & -\,0.10413b_4 & = & -\,0.02340 \\
\text{(II)} & -\,394.15b_2 & +\,676.55b_3 & +\,11.60b_4 & = & 1360.60 \\
(0.64939)\text{(I)} & 394.15b_2 & -\,255.96b_3 & +\,41.04b_4 & = & 9.22 \\
\hline
(\Sigma_2) & & 420.59b_3 & +\,52.64b_4 & = & 1369.82 \\
\text{(II}'\text{)} & & -\,b_3 & -\,0.12516b_4 & = & -\,3.25690 \\
\text{(III)} & 63.20b_2 & +\,11.60b_3 & +\,17.20b_4 & = & 193.20 \\
(-0.10413)\text{(I)} & -\,63.20b_2 & +\,41.04b_3 & -\,6.58b_4 & = & -\,1.48 \\
(-\,0.12516)\ (\Sigma_2) & & -\,52.64b_3 & -\,6.59b_4 & = & -\,171.45 \\
\hline
(\Sigma_3) & & & 4.03b_4 & = & 20.27 \\
\text{(III}'\text{)} & & & -\,b_4 & = & -\,5.02978
\end{array}
$$

It is now very easy to compute the values of b_2, b_3, and b_4, from the three derived equations. From equation (III′), $b_4 = 5.02978$.

Substituting this value in equation (II′), which may be transposed to read

$$b_3 = 3.25690 - 0.12516b_4$$

we find

$$b_3 = 3.25690 - (0.12516)(5.02978)$$
$$= 3.25690 - 0.62953 = 2.62737$$

Then, transposing equation (I′),

$$b_2 = 0.02340 + 0.64939b_3 - 0.10413b_4,$$

and substituting the values for b_3 and b_4,

$$b_2 = 0.02340 + (1.70619) - (0.52375),$$

we find

$$b_2 = 1.20584$$

The values of b_2, b_3, and b_4, just computed, may next be verified by substituting them in the last equation (III). *Equations* (I) *or* (II) *should not be used for this verification, since they will not provide a complete check.* Equation (III)

$$63.20b_2 + 11.60b_3 + 17.20b_4 = 193.20$$

becomes, when the newly-calculated values are substituted,

$$(63.20)(1.20584) + (11.60)(2.62737) + (17.20)(5.02978) = 193.20;$$

this works out to

$$76.21 + 30.48 + 86.51 = 193.20$$

or

$$193.20 = 193.20$$

This proves the accuracy of all the previous work.

The work shown on the last page is all that is needed to solve these three simultaneous equations. In view of the way the terms cancel out during the second and subsequent steps of the process, the work can be still further simplified by omitting all entries to the left of the solid line which has been drawn in through the last set of entries.

Having calculated the values of the three b's, a is very readily calculated.

$$\begin{aligned} a &= M_1 - b_2M_2 - b_3M_3 - b_4M_4 \\ &= 87.2 - (1.20584)(13.95) - (2.62737)(8.85) - (5.02978)(2.20) \\ &= 36.06 \end{aligned}$$

The regression equation for the three variables is therefore

$$\left(\frac{X_1}{10}\right) = 36.06 + 1.20584\left(\frac{X_2}{10}\right) + 2.62737X_3 + 5.02978X_4$$

Clearing the fractions, the equation becomes

$$X_1 = 360.60 + 1.20584X_2 + 26.2737X_3 + 50.2978X_4$$

Using this equation, values of X'_1 and of z may be worked out just as was done previously. (This will be left as an exercise for the student. Is σ_z for the new estimates larger or smaller than for the previous estimates? Why should it be?)

Interpreting net regression coefficients.—It should be noted that though the value of 1.20584 for $b_{12.34}$, just determined, compares with the value of 2.13840, for $b_{12.3}$ determined previously, they do not measure exactly the same thing. The coefficient $b_{12.34}$ shows the average increase in income for each increase of one acre in size of farm, with both the number of *cows* and the number of *men* remaining unchanged. The coefficient $b_{12.3}$ showed the average increase in income for each increase of one acre in size, with the number of *cows* remaining unchanged, but without making any allowance for differences in the number of men. Apparently a considerable portion of the differences in income which on the earlier analysis would have been ascribed to the additional acreage is shown by this more complete analysis to have been really associated with the larger labor force on the greater acreages, rather than to the greater acreages themselves. This result illustrates one property of net regression coefficients in common with all other correlation results. They ascribe to any particular independent variable not only the variation in the dependent variable which is directly due to that independent variable, but also the variation which is due to such other independent variables correlated with it as have not been separately considered in the study. In the same way that acres, taken alone, included part of the effect due to cows, the effect of acres eliminating cows still included part of the effect due to men; and even the effect of acres holding constant the effect of both cows and men may still include variation due to other correlated variables, such, for example, as fertility of the land. These considerations illustrate the extreme care which is necessary in examination of the data and the theoretical analysis of the problem before deciding on the variables to be correlated, and the caution which must be employed in interpreting the results.

Determining the Regression Equation for Any Number of Independent Variables.—The same mathematical principle which has been used to determine the constants for regression equations involving one, two, or three independent variables can be extended to problems involving any number of variables it may be desired to employ.

For 4 independent variables the equations are:

$$\left.\begin{aligned}
&\Sigma(x_2^2)b_{12.345} + \Sigma(x_2x_3)b_{13.245} + \Sigma(x_2x_4)b_{14.235} \\
&\qquad + \Sigma(x_2x_5)b_{15.234} = \Sigma(x_1x_2) \\
&\Sigma(x_2x_3)b_{12.345} + \Sigma(x_3^2)b_{13.245} + \Sigma(x_3x_4)b_{14.235} \\
&\qquad + \Sigma(x_3x_5)b_{15.234} = \Sigma(x_1x_3) \\
&\Sigma(x_2x_4)b_{12.345} + \Sigma(x_3x_4)b_{13.245} + \Sigma(x_4^2)b_{14.235} \\
&\qquad + \Sigma(x_4x_5)b_{15.234} = \Sigma(x_1x_4) \\
&\Sigma(x_2x_5)b_{12.345} + \Sigma(x_3x_5)b_{13.245} + \Sigma(x_4x_5)b_{14.235} \\
&\qquad + \Sigma(x_5^2)b_{15.234} = \Sigma(x_1x_5)
\end{aligned}\right\} \tag{40}$$

$$a_{1.2345} = M_1 - b_{12.345}M_2 - b_{13.245}M_3 - b_{14.235}M_4 - b_{15.234}M_5 \tag{41}$$

Comparing this set of equations with equation (38) for three independent variables, it is evident that adding the additional variable, X_5, has made it necessary to add the additional equation, in which X_5 appears in each of the product terms, and also to add an additional term to each of the previous equations, the additional term including a product summation (such as $\Sigma(x_2x_5)$, $\Sigma(x_3x_5)$, etc.) in which X_5 appears, and also the net regression coefficient $b_{15.234}$. The equation to compute a has also been extended by adding the term "$-b_{15.234}M_5$." In the same way the equations to be solved to determine the constants for any number of variables can be built up, remembering that for each variable added a new term must be added to each of the previous equations, and a new equation must be added, each term added including the new variable in some way.

The products which must be computed for any given set of variables, and the equations which will need to be solved, may be worked out readily by the use of the following scheme:

Write out the required regression equation (in terms of deviations from the mean), as, for example, for six variables:

$$b_2x_2 + b_3x_3 + b_4x_4 + b_5x_5 + b_6x_6 = x_1$$

Multiply each term through by the coefficient of the first unknown (that is, by x_2), and sum. This gives the first of the required equations:

$$\Sigma(x_2^2)b_2 + \Sigma(x_2x_3)b_3 + \Sigma(x_2x_4)b_4 + \Sigma(x_2x_5)b_5 + \Sigma(x_2x_6)b_6 = \Sigma(x_2x_1)$$

Then multiply through by the coefficient of the second unknown (x_3) and sum. The second equation is therefore

$$\Sigma(x_2x_3)b_2 + \Sigma(x_3^2)b_3 + \Sigma(x_3x_4)b_4 + \Sigma(x_3x_5)b_5 + \Sigma(x_3x_6)b_6 = \Sigma(x_3x_1)$$

The same process is carried out for the coefficient of each unknown in turn, giving five equations to be solved simultaneously to determine the values for the five unknowns. Setting up these equations may be reduced to a tabular form, as follows:

TABLE 48

FORM FOR WORKING OUT THE EQUATIONS TO DERIVE NET REGRESSION CONSTANTS

Independent variables	Independent variables (in deviations from means)							Dependent variable x_1
	x_2	x_3	x_4	x_5	x_6	x_7	x_8	
x_2	$\Sigma(x_2^2)b_2$	$\Sigma(x_2x_3)b_3$	$\Sigma(x_2x_4)b_4$					$=\Sigma(x_1x_2)$
x_3	$\Sigma(x_2x_3)b_2$	$\Sigma(x_3^2)b_3$	$\Sigma(x_3x_4)b_4$					$=\Sigma(x_1x_3)$
x_4	$\Sigma(x_2x_4)b_2$	$\Sigma(x_3x_4)b_3$	$\Sigma(x_4^2)b_4$					$=\Sigma(x_1x_4)$
x_5	$\Sigma(x_2x_5)b_2$	$\Sigma(x_3x_5)b_3$	$\Sigma(x_4x_5)b_4$					$=\Sigma(x_1x_5)$
x_6	$\Sigma(x_2x_6)b_2$	$\Sigma(x_3x_6)b_3$	$\Sigma(x_4x_6)b_4$					$=\Sigma(x_1x_6)$
x_7	$\Sigma(x_2x_7)b_2$	$\Sigma(x_3x_7)b_3$	$\Sigma(x_4x_7)b_4$					$=\Sigma(x_1x_7)$
x_8	$\Sigma(x_2x_8)b_2$	$\Sigma(x_3x_8)b_3$	$\Sigma(x_4x_8)b_4$					$=\Sigma(x_1x_8)$

The variables to be considered are listed at the head of columns from the left to right, ending with the dependent variable at the right. Then the independent variables are entered down the beginning of the lines at the left in the same order. The cells of the table are then filled by multiplying the variable at the head of the column by the variable at the end of the line. These products indicate the values to be computed (by equations 11 and 15), to give the arithmetic values for the equations. The "b" terms represent, of course, the net regression coefficients for the particular number of variables concerned; that is, b_2 would be $b_{12.3}$ for two independent variables, $b_{12.34}$ for three independent variables, and so on. The illustration is carried out to seven independent variables, but the scheme can be extended to as many as it is desired to consider.

The equation to compute a is simply the value of the mean of the dependent variable, minus the product of the mean of each independent variable multiplied by the coefficient for the net regression of the dependent variable on that independent variable.

As a matter of practical procedure, it is seldom that a problem is so complicated or that enough observations are available, so that significant results for each variable will be obtained using ten or more variables; and, ordinarily, analyses involving not more than five variables are all that will yield stable results. To illustrate some of the details of the procedure necessary where a large number of variables must be considered, various methods to simplify the necessary calculations in carrying through a problem involving a large number of observations are presented in the Appendix on Methods of Computation.

Interpreting the Multiple Regression Equation.—The same limitations apply in interpreting regression coefficients worked out with the effect of one or more variables held constant as when only two variables are considered. Thus for the data shown in Table 47: there were no observations with more than 18 cows, or 4 men, and none below 60 acres or above 240 acres. For that reason, there is no basis for using the regression equation to estimate income beyond those limits. Furthermore, for the extreme ranges where only a few observations were available—for example, less than 80 acres—the relations could not be expected to hold as well as where there were more observations upon which to base the conclusions. In Chapter 18 a more definite basis for determining the probable accuracy of such estimates is discussed. For the present the caution may be re-stated, that the results may be expected to hold true only within the range covered by the bulk of the observations upon which they were based.

The meaning of the regression equation

$$X_1 = 360.60 + 1.21X_2 + 26.27X_3 + 50.30X_4$$

may be made clearer, in publishing correlation results, by working out the estimated values for a representative variety of conditions. Such a statement of the conclusions covered by the previous regression equation would be as follows:

TABLE 49

Average Income on Farms With Varying Numbers of Acres, Cows, and Men

(As indicated by correlation analysis)

Labor force	100 acres			160 acres		
	0 cows	8 cows	16 cows	0 cows	8 cows	16 cows
	Dollars	*Dollars*	*Dollars*	*Dollars*	*Dollars*	*Dollars*
1 man......	532	742	952	604	815	1023
2 men......	582	792	1003	655	865	1075
3 men......	632	843	1053	705	915	1125

It should be noted in Table 49 that according to these results, increasing the number of men from 1 to 2, or from 2 to 3, will add $50 to income, no matter whether the farm has 100 acres and no cows, or 160 acres and 16 cows. Similarly, adding 8 more cows is indicated as having the same effect on income, no matter how large the farm is or how many men are employed. But that this conclusion has been reached is no proof that it is really true of the universe represented by the original data. Instead, such a conclusion is inherent in the linear equation (35, 36, or 37) which has been used. That equation necessarily assumes that an increase of one unit in any one independent variable will always be accompanied by an equal change in the dependent variable. Only insofar as the actual facts agree with that assumption can they be represented by a linear equation. Subsequent chapters (particularly 14 and 20) take up methods of analysis which may be employed when this type of relation is not true, and the linear equation is therefore unable to express the facts adequately.

Summary.—This chapter has presented mathematical methods for determining the constants of a linear regression equation, so that changes in one variable may be estimated from changes in two or more independent variables. Equations so determined afford a more exact basis for making such estimates than do linear equations obtained by any other method. Furthermore, the multiple regression equation serves to sum up all the evidence of a large number of observations in a single statement which expresses in condensed form the extent to which differences in the dependent variable tend to be associated with differences in each of the other variables.

CHAPTER 13

MEASURING ACCURACY OF ESTIMATE AND DEGREE OF CORRELATION FOR LINEAR MULTIPLE CORRELATION

Standard error of estimate.—After working out equations by which values of one variable may be estimated from those for two or more independent variables, it is frequently desirable to have some measure of how closely such estimates agree with the actual values, and of how closely the variation in the dependent variable is associated with the variation in the several independent variables. Attention has been called in the preceding chapters to the computation of the residual variation, z, when the value of a variable is estimated from that of several others. Where the estimate is based on several independent variables the standard deviation of these residuals serves as a measure of the closeness with which the original values may be estimated or reproduced just as well as where it is based on a single variable. Continuing the same terminology as before, this standard deviation is still called the "standard error of estimate." Thus for the regression equation for estimating income from known numbers of acres, cows, and men, the standard error of estimate is designated $S_{1.234}$. The subscripts "$_{1.234}$" indicate that that is the standard error for variable X_1 when estimated from the independent variables X_2, X_3, and X_4.

Where the size of the sample is small in proportion to the number of variables involved, the standard deviation of the residuals for the cases included in the sample tends to be smaller than the true standard error of estimate—that is, smaller than the standard deviation of the errors which will be made when the regression equation is employed to estimate values of the dependent factor for new cases drawn from the same universe. For that reason it is necessary to adjust the observed standard deviation of the residuals, σ_z, before it will give the true value for the standard error of estimate. This adjustment, which should be made whenever the value of the standard error is changed by making it, is as follows:

$$\bar{S}^2_{1.234} = \frac{n\sigma^2_{z_{1.234}}}{n - m} \qquad (42)$$

Where n = number of sets of observations in the sample,
m = number of constants in the regression equation, including a and the b's.

(Where the adjusted value for $\bar{S}^2_{1.234}$ exceeds the value of σ_1^2, the latter value should be used for the standard error as well.)

The standard errors for the equations obtained when one, two, and three independent variables were considered in the farm income study in Chapter 12 may be summarized as follows:

Independent variables	Observed σ_z	n	m	Standard error
X_2	165.15*	20	2	$\bar{S}_{1.2} = 165.15$
X_2, X_3	70.48	20	3	$\bar{S}_{1.23} = 76.45$
X_2, X_3, X_4	66.77	20	4	$\bar{S}_{1.234} = 74.65$

* This value has not been shown previously. It is calculated from the data of Chapter 12.

This illustrates the increase in the reliability of estimate as additional variables are taken into account.

So far, the standard errors of estimate (except for simple or two-variable correlation) have been determined by actually working out all the estimated values, subtracting to get the individual residuals, z, and then determining their standard deviation. For linear multiple regression equations, however, a much simpler process can be used. To compute the standard deviation of the residuals by this process, all that is required in addition to the values which have been used in computing the b's is the value, $\Sigma(x_1^2)$. The formula is as follows:

$$\bar{S}^2_{1.234\ldots n} = \frac{\Sigma(x_1^2) - [b_{12.34\ldots n}(\Sigma x_1x_2) + b_{13.24\ldots n}(\Sigma x_1x_3) + \ldots]}{n - m} \qquad (43)$$

Substituting the values for the regression equation computed with two independent variables, pages 161 and 162, the equation becomes

$$\bar{S}^2_{1.23} = \frac{\Sigma(x_1^2) - [b_{12.3}(\Sigma x_1x_2) + b_{13.2}(\Sigma x_1x_3)]}{n - 3}$$

In terms of coded values for X_1,

$$\frac{\bar{S}^2_{1.23}}{10} = \frac{5455.20 - (2.1384)(14.20) - (3.2569)(1360.60)}{20 - 3}$$

$$\frac{\bar{S}_{1.23}}{10} = \sqrt{\frac{993.50}{17}} = 7.645; \quad \bar{S}_{1.23} = 76.45$$

The result is seen to be identical with the value computed (after adjustment) by the lengthy process illustrated in Table 46, of working out all the individual estimates, computing their standard deviation, and then correcting by equation (42).

Multiple Correlation.—The standard error of estimate for a multiple regression equation, just as with simple correlation, measures the absolute accuracy with which the estimated values agree with the original values. The standard error, however, offers no measure of the *proportion* of the variation in the dependent factor which can be explained by, or is associated with, variation in the independent factor or factors. For example, in one area the farm income might be twice as variable as in another. If two or three independent factors such as those discussed came as near accounting for all the variation in incomes in one area as in the other, the standard errors of estimate would be the same in both cases. There was originally more variance in income in the one case than in the other; therefore with the same amount left unaccounted for the independent factors would have been associated with a larger proportion of the original variance, in the case where it was largest to begin with, and would have been relatively more important in that case. In simple correlation, the *relative* importance of the independent factor was measured by the ratio of the standard deviation of the estimated values to the standard deviation of the actual values, and the name *coefficient of correlation* was given to this ratio. In exactly similar manner, when the estimates are based on several variables, instead of on one, the relative importance of all of those variables combined may be measured by dividing the standard deviation of the estimated values by that of the original values. This ratio is named the *coefficient of multiple correlation,* since it measures the combined importance of the several independent factors as a means of explaining the differences in the dependent factor.

If we use $X_{1(234)}$ to designate the estimates of X_1 made from variables X_2, X_3, and X_4, and use $R_{1.234}$, to represent the unadjusted *coefficient of multiple correlation*, the coefficient may be defined:

$$X_{1(234)} = a_{1.234} + b_{12.34}X_2 + b_{13.24}X_3 + b_{14.23}X_4 \tag{44}$$

$$R_{1.234} = \frac{\sigma_{1(234)}}{\sigma_1} \tag{45}$$

The same short formula which has been shown for computing the standard error of estimate may be employed to facilitate the computation of the coefficient of multiple correlation, using only values already involved in equation (43). The equation for computing the coefficient of correlation by this method is as follows:

$$R^2_{1.234\ldots n} = \frac{\left\{ \begin{array}{c} b_{12\cdot 34\ldots n}(\Sigma x_1x_2) + b_{13\cdot 24\ldots n}(\Sigma x_1x_3) \\ + \ldots + b_{1n\cdot 23\ldots (n-1)}(\Sigma x_1x_n) \end{array} \right\}}{\Sigma(x_1^2)} \qquad (46)$$

There is a tendency for the multiple correlation shown by the sample to be in excess of the correlation existing in the universe from which the sample was drawn, especially where the number of observations is small, or the number of variables large. For that reason the coefficient $R_{1.23\ldots n}$, computed as shown in equation (46), has to be adjusted before it will give $\bar{R}_{1.23\ldots n}$, the correlation most probably existing in the whole universe. The adjustment is as follows:

$$\bar{R}^2_{1.234\ldots n} = 1 - (1 - R^2_{1.234})\left(\frac{n-1}{n-m}\right) \qquad (47)$$

m and n have the same meaning for this equation as in equation (42).

If the value for $\bar{R}^2$ comes out a minus quantity, then use 0 for $\bar{R}^2$.

The square of the coefficient of multiple correlation, $\bar{R}^2$, may be termed the *coefficient of multiple determination.*

The same relations hold between the coefficient of multiple correlation and the standard error of estimate in the case of multiple correlation as in the case of simple correlation. For that reason, one of these measures may be computed from the other, whichever is determined first, according to the following equations:

$$\bar{R}^2_{1.234\ldots n} = 1 - \left(\frac{\bar{S}^2_{1.234\ldots n}}{\sigma_1^2}\right)\left(\frac{n-1}{n}\right) \qquad (48)$$

$$\bar{S}^2_{1.234\ldots n} = \sigma_1^2(1 - \bar{R}^2_{1.234\ldots n})\left(\frac{n}{n-1}\right) \qquad (49)$$

Using equation (48) to compute the values of $\bar{R}$ from the values of $\bar{S}$ previously computed, the multiple coefficients for the three regression equations previously worked out may be stated in the following different ways:

Dependent variable	Independent variable(s)	Standard error of estimate	Coefficient of multiple correlation	Coefficient of total determination
X_1(income)	X_2(acres)	165.15	0	0
X_1(income)	X_2(acres); X_3(cows)	76.45	0.892	0.796
X_1(income)	X_2(acres); X_3(cows); X_4(men)	74.65	0.898	0.806

It is evident that the correlation increases as the standard error decreases. Here the residual variation in each case is being compared with the same original standard deviation, so that that necessarily follows. Where different studies are being compared, however, such as two samples with widely different original deviations in the dependent variable, the standard error of estimate would not necessarily decrease as the correlation increased, since the former is an *absolute* measure, whereas the latter is a *relative* measure.[1]

It is evident from the figures just shown that the coefficient of multiple correlation, if incorrectly interpreted, makes the relationship seem closer than does the coefficient of multiple determination (R^2). It cannot be demonstrated that the coefficient of multiple determination will measure in all cases that proportion of the variance in the dependent factor which is associated with the independent factors. Yet it is sufficiently true so that if a statement is to be made such as, "seventy-five per cent of the variance in income was associated with (or related to) variances in numbers of acres farmed, or cows milked, and men hired," it is more accurate to use the coefficient of multiple determination than to use the coefficient of multiple correlation. The latter would overstate the case. This principle holds true both for simple correlation ($\bar{r}$) and multiple correlation ($\bar{R}$): the square of the coefficient indicates the proportion of the variance in the dependent variables which has been mathematically accounted for; whereas one minus the square of the coefficient indicates the proportion which has not been accounted for.[2]

The coefficient of multiple correlation, $R_{1.234\ldots n}$, may also be defined as the simple correlation between the actual X_1 values and the $X_{1(234)}$ values estimated from the several independent factors. This interpretation illustrates the way it sums up the combined relation of the dependent variable to the several independent variables.

Measuring the Separate Effect of Individual Variables.—In addition to the measures of the importance of all of the independent variables combined, it is sometimes desirable to have measures of the importance of each of the individual variables taken separately, while simultaneously allowing for the variation associated with remaining independent variables. There are three different types of these measures: *the coefficient of partial correlation, the coefficient of part correlation,* and the *"beta" coefficient.*

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[1] This point is of considerable significance in certain types of economic problems, particularly in time series analysis. For example, taking the first differences of a series of values frequently tends to make the deviations much larger than by taking deviations from trend. A study which gives a higher coefficient of correlation for first differences than for deviations from trend, may still yield the less accurate estimate, as measured by the standard error of estimate.

[2] See Technical Appendix, Note 7.

Partial correlation.—Coefficients of *partial correlation* serve to determine the correlation between the dependent factor and each of the several independent factors, while eliminating any (linear) tendency of the remaining independent factors to obscure the relation. Thus in the problem where income was correlated with numbers of acres, cows, and men, the partial correlation of income with acres, while holding constant cows and men, indicates what the average correlation would probably be between acres and income in samples of farms in which all the farms in each sample had the same number of cows and the same number of men.

Any group of independent variables may serve to explain some, but not all, of the variation in a dependent variable. If an additional independent variable is added, it may account for part of the variation left unexplained by the factors previously considered. The coefficient of partial correlation may be defined as a measure of the extent to which that part of the variation in the dependent variable which was *not* explained by the other independent factors can be explained by the addition of the new factor. For example, in the farm income problem, considering only acres and cows, the correlation was $\bar{R}_{1.23} = 0.892$. When acres, cows, and men were considered, the correlation was $\bar{R}_{1.234} = 0.898$. Squaring both values shows that while the two variables explain 79. 6 per cent of the variance in income, the three variables explain 80.6 per cent. Whereas 20.4 per cent of the variance is left to be explained when the two variables are considered, only 19.4 per cent is left to be explained when three are considered. Adding the additional variable has increased the variance which can be explained by the difference between these two figures, or 1.0 per cent (20.4 − 19.4 per cent). If the importance of this increase is determined by comparing it to the variance left unexplained before the new variable was added, we find that $\frac{1.0}{20.4}$, or 4.90 per cent of the variance left unexplained by acres and cows, has now been found to have been associated with differences in numbers of men. Taking its square root gives the coefficient of partial correlation, 0.221.

The coefficient is designated $\bar{r}_{14.23}$, since it shows the partial correlation between X_1 and X_4, after X_2 and X_3 had been taken into account.

As is indicated in the discussion, it may be computed by the formula[3]

$$\bar{r}^2_{14.23} = \frac{(1 - \bar{R}^2_{1.23}) - (1 - \bar{R}^2_{1.234})}{1 - \bar{R}^2_{1.23}}$$

For purposes of computation, this formula may be simplified to

$$\bar{r}^2_{14.23} = 1 - \frac{(1 - \bar{R}^2_{1.234})}{(1 - \bar{R}^2_{1.23})} \qquad (50)$$

If it is desired to compute coefficients of partial correlation for the other independent variables, acres and cows, the corresponding formulas are

$$\bar{r}^2_{13.24} = 1 - \frac{(1 - \bar{R}^2_{1.234})}{(1 - \bar{R}^2_{1.24})}$$

$$\bar{r}^3_{12.34} = 1 - \frac{(1 - \bar{R}^2_{1.234})}{(1 - \bar{R}^2_{1.34})}$$

It should be noticed that although the upper portion of the fraction is the same in each case, the lower part is different. This is a peculiarity of coefficients of partial correlation—they measure the importance of each of the several variables by determining how much it reduces the variation *after all the other variables except it are taken into account.*

Working out the new multiple correlations necessary,[4] $\bar{R}_{1.24}$ and $\bar{R}_{1.34}$, and substituting in the equations given just above, the whole set of coefficients of partial correlation and partial determination for the farm income problem work out as follows:

$$\bar{r}^2_{13.24} = 1 - \frac{(1 - 0.806)}{(1 - 0.458)} = 0.642$$

$$\bar{r}^2_{12.34} = 1 - \frac{(1 - 0.806)}{(1 - 0.791)} = 0.072$$

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[3] This is different from the formula customarily given. See Technical Appendix, Note 7, for its derivation.

[4] The two new coefficients of multiple correlation are obtained by rearranging the arithmetic values previously computed so as to give the necessary regression coefficients, and then determining the value of R by equation (46). The two new sets of equations are as follows:

To determine $R_{1.24}$

$$(\Sigma x_2^2)b_{12.4} + (\Sigma x_2 x_4)b_{14.2} = (\Sigma x_1 x_2)$$
$$(\Sigma x_2 x_4)b_{12.4} + (\Sigma x_4^2)b_{14.2} = (\Sigma x_1 x_4)$$

Similarly for $R_{1.34}$

$$(\Sigma x_3^2)b_{13.4} + (\Sigma x_3 x_4)b_{14.3} = (\Sigma x_1 x_3)$$
$$(\Sigma x_3 x_4)b_{13.4} + (\Sigma x_4^2)b_{14.3} = (\Sigma x_1 x_4)$$

Relative Importance of Individual Factors Affecting Income, as Indicated, by Coefficients of Partial Correlation

Factors already considered	Factor added	Coefficient of partial correlation ($\bar{r}_{12.34}$, etc.)	Reduction in unexplained variance ($\bar{r}^2_{12.34}$, etc.)
Cows (X_3), men (X_4).....	Acres (X_2)	0.27	0.072
Acres (X_2), men (X_4).....	Cows (X_3)	0.80	0.642
Acres (X_2), cows (X_3).....	Men (X_4)	0.22	0.049

When income was correlated with acres alone, there was no correlation at all. (Before adjusting for the number of variables, $r_{12} = 0.01$.) Yet the partial correlation of income with acres, while holding constant the variation associated with cows and men, has just been seen to be 0.27. Although this is not high, it is certainly more than no correlation at all. Furthermore, even though the correlation of income with cows alone is 0.64, the correlation with both acres and cows is 0.89.

On the surface of the data there appears to be no relation between acres and income, since the positive relation of acres to income is hidden. Acres are negatively correlated with cows to a sufficient extent so that the decreased income with decreased number of cows offsets the increases with more acres. Only when the number of cows is allowed for can the influence of acres be seen.

It is evident that a mere surface examination of a set of data cannot reveal which independent factors are important and which are unimportant. A variable which shows no correlation with the dependent variable may yet show significant correlation after the relation to other variables has been allowed for.

The test as to whether a given independent variable may really be related to the dependent variable, even if it shows no apparent correlation, is whether that independent variable is correlated with other independent variables which in turn are correlated with the dependent. Thus in the example just discussed, although acres showed no correlation with income, they did show significant correlation with cows. If acres had had no correlation with either income, cows, or men, then it would have been impossible for acres to have correlation with income even after the relation to cows and men was allowed for.

Part correlation.—Measuring the separate importance of each of the factors by partial correlation involves computing as many additional coefficients of multiple correlation as there are independent variables. Somewhat less work is involved in the computation of an alternative measure, the coefficient of *part correlation.* This coefficient measures the correlation between the dependent factor, after removing the net variations found to be associated with the remaining independent factors, and the particular independent factors to be considered. Thus in the farm income problem, with three independent factors, the regression equation was

$$X_1 = a_{1.234} + b_{12.34}X_2 + b_{13.24}X_3 + b_{14.23}X_4$$

If X_1 were adjusted by subtracting from it the variation ascribed to X_3 and X_4, the adjusted value would be

$$X_1 - (b_{13.24}X_3 + b_{14.23}X_4)$$

Then if X_2 were correlated with this adjusted value, the resulting correlation would indicate how closely the variation in X_1 was associated with X_2, after the final effects of X_3 and X_4 had been removed from X_1. This coefficient may be designated the *coefficient of part correlation* between X_1 and X_2, with the effects of X_3 and X_4 removed. To distinguish it from the coefficient of partial correlation, it may be represented by the symbol "${}_{12}r_{34}$"; the subscripts to the left representing, first, the dependent variable, the next, the independent variable whose effect is measured, and the subscripts to the right representing the independent variables whose effects are removed.[5]

Figures 28 and 29 in Chapter 10 indicate roughly the type of relation measured in part correlation. In these dot charts the income, adjusted for the net effects of one factor, is plotted against the other factor.

To compute coefficients of part correlation by the method indicated by their definition would be a laborious task, but they can be readily computed by use of the formula [6]

$${}_{12}\bar{r}^2{}_{34} = \frac{b^2{}_{12.34}\sigma_2{}^2}{b^2{}_{12.34}\sigma_2{}^2 + \sigma_1{}^2(1 - \bar{R}^2{}_{1.234})} \qquad (51)$$

This formula involves no values other than those worked out in obtaining the multiple correlation of X_1 with X_2, X_3, and X_4, and but little additional work is required to compute the part correlation.

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[5] It should be noted that though with coefficients of *partial correlation*, $r_{12.34} = r_{21.34}$, with coefficients of *part correlation*, ${}_{12}r_{34}$ is *not* equal to ${}_{21}r_{34}$. Care should therefore be taken in designating these latter coefficients.

[6] See Technical Appendix, Note 9, for the derivation of this formula.

Thus, substituting the values for the farm income problem, the equation becomes

$$ {}_{12}\bar{r}^2{}_{34} = \frac{(1.454)(30.35)}{(1.454)(30.35) + (272.76)(0.196)} = \frac{44.13}{97.59} = 0.452 $$

The corresponding equations for the other variables are

$$ {}_{13}\bar{r}^2{}_{24} = \frac{b^2{}_{13.24}\sigma_3{}^2}{(b^2{}_{13.24}\sigma_3{}^2) + \sigma_1{}^2(1 - \bar{R}^2{}_{1.234})} $$

$$ {}_{13}\bar{r}^2{}_{24} = 0.814 $$

$$ {}_{14}\bar{r}^2{}_{23} = \frac{b^2{}_{14.23}\sigma_4{}^2}{(b^2{}_{14.23}\sigma_4{}^2) + \sigma_1{}^2(1 - \bar{R}^2{}_{1.234})} $$

$$ {}_{14}\bar{r}^2{}_{23} = 0.289 $$

Working these out, the part correlation of farm income with acres is found to be 0.67; with cows, 0.90; and with men, 0.54, with the influence of the other two factors removed in each case. Taking the squares of these values as the *coefficients of part determination*, they may be explained as measuring the proportion of the variance remaining in X_1 after the net effects of the other variables are taken account of, which can be explained by adding the additional factor. In this case they are, for X_1 and X_2, 45.2 per cent; for X_1 and X_3, 81.4 per cent; and for X_1 and X_4, 28.9 per cent.[7]

"Beta" coefficients.—Besides the two ways of comparing the importance of individual variables, they may also be compared by their regression coefficients. The size of the regression coefficients, however, varies with the units in which each variable is stated. They may be made more comparable by expressing each variable in terms of its own standard deviation, using the "beta" coefficients mentioned in Chapter 9. In terms of betas, the regression equation for four variables would be

$$ \frac{X_1}{\sigma_1} = \beta_{12.34}\frac{X_2}{\sigma_2} + \beta_{13.24}\frac{X_3}{\sigma_3} + \beta_{14.23}\frac{X_4}{\sigma_4} + a' $$

Hence the partial betas may be defined

$$ \beta_{12.34} = b_{12.34}\frac{\sigma_2}{\sigma_1} \tag{52} $$

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[7] The difference between the coefficients of part correlation and of partial correlation may be specifically stated as follows, in terms of a 4-variable problem:

${}_{12}r_{34}$ is the correlation of X_2 with $[X_1 - b_{13.24}X_3 - b_{14.23}X_4]$.

$r_{12.34}$ is the correlation of $[X_2 - b_{23.4}X_3 - b_{24.3}X_4]$ with $[X_1 - b_{13.4}X_3 - b_{14.3}X_4]$

Of the two, the part correlation is the simpler to understand and the easier to compute.

For the problem we have been considering, the betas may be calculated very readily:

$$\beta_{12.34} = b_{12.34}\frac{\sigma_2}{\sigma_1} = 1.2058\left(\frac{5.51}{16.52}\right) = 0.402$$

$$\beta_{13.24} = b_{13.24}\frac{\sigma_3}{\sigma_1} = 2.6274\left(\frac{5.82}{16.52}\right) = 0.926$$

$$\beta_{14.23} = b_{14.23}\frac{\sigma_4}{\sigma_1} = 5.0298\left(\frac{0.927}{16.52}\right) = 0.282$$

If the relative importance of each of the different factors, as judged by the three different types of individual measurement, is compared, the relations are as follows:

RELATIVE IMPORTANCE OF INDIVIDUAL FACTORS AFFECTING INCOME, AS INDICATED BY THREE DIFFERENT COEFFICIENTS

Independent factor	Factors held constant	Coefficients of		Beta coefficients $\beta_{12.34}$
		Partial correlation ($r_{12.34}$)	Part correlation ($_{12}r_{34}$)	
Acres (X_2).......	Cows (X_3), men (X_4)	0.27	0.67	0.402
Cows (X_3).......	Acres (X_2), men (X_4)	0.80	0.90	0.926
Men (X_4).......	Acres (X_2), cows (X_3)	0.22	0.54	0.282

It is evident from this comparison that although the exact values differ for each of the three sets of measures, the rank of the three variables in order of importance is the same, and the relative sizes are comparable.[8] This does not always hold true, owing to the mathematical differences in the meaning of the three sets. The partial correlation coefficients, though the earliest in the development of multiple correlation method for biometric problems, are both complicated to work out and difficult to interpret. They have been presented here largely so that they might be understood when used. It is not believed that they are especially superior to coefficients of "part" correlation; in view of the ease of computation of the latter, they would ordinarily be preferred.[9]

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[8] One other type of measure of individual importance, the coefficient of separate determination, is discussed in Note 11, of the Technical Appendix.

[9] Although this seems true for economic problems, the coefficients of partial correlation may be more valuable for biometric problems. See Technical Appendix, Note 10.

Besides the coefficients which have been discussed, which measure either the total relative importance of all the independent variables, or the importance of each one separately, it is sometimes desirable to measure the correlation between one variable and a group of others, after eliminating from the dependent variable that part of its variation imputed (by the analysis) to a single one of the independent variables. The problem may be stated as follows:

Where $R_{1 \cdot 234}$ measures the relation between X_1 and X_2, X_3, X_4, according to the regression equation (36) the problem stated is to determine the correlation between $(X_1 - b_{12.34}X_2)$ and the two remaining independent variables, according to the equation

$$(X_1 - b_{12.34}X_2) = a_{1.234} + b_{13.24}X_3 + b_{14.23}X_4$$

This could be determined by actually carrying out the operations indicated, but it can be much more readily computed by use of the formula[10]

$$\left.\begin{array}{c}\text{Multiple correlation}\\ \text{squared of}\\ (X_1 - b_{12.34}X_2)\\ \text{with } X_3 \text{ and } X_4\end{array}\right\} = 1 - \frac{\sigma_1{}^2(1 - R^2{}_{12.34})}{\sigma_1{}^2 - 2b_{12.34}(\Sigma x_1x_2/n) + b^2{}_{12.34}\sigma_2{}^2} \qquad (53)$$

An illustration of the type of problem to which this method may be applied can be drawn from the field of price analysis. If X_2 in the case illustrated above were an index of price level, X_1 the price of some commodity, and X_3 and X_4 other factors affecting price, such as production and storage stocks, it might be desired to determine not only how closely the price of the commodity was related to all the factors, including the price index, but also how closely it was related to the remaining factors after the variations in price found to be associated with changes in price level were removed from it. Formula (53) would enable this determination to be made.

Summary.—This chapter has shown that the absolute accuracy of a regression equation for estimating one variable from two or more others may be measured by the standard error of estimate. The extent to which variation in the dependent variable is associated with the variation in the several independent variables may be measured by the coefficient of multiple correlation, or, with respect to variance, by the coefficient of total determination. The relative importance of each of the independent variables may be measured (a) by the coefficient of partial correlation, relative to the variation remaining after the effects of the other variables have first been removed; or (b) by the coefficient of part correlation, relative to the variation remaining while the final net effects of the other variables are removed simultaneously; or (c) by the beta coefficients, which reduce the net regression coefficients to a comparable basis. Finally, a method is provided for measuring the proportion of the variation in the dependent variable which is explainable by a group of independent variables, after eliminating from the dependent variable that portion of its variability which has been found to be associated with another independent factor.

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[10] See Technical Note 12 for derivation of this equation.

Farris part correlation.

CHAPTER 14

DETERMINING THE WAY ONE VARIABLE CHANGES WHEN TWO OR MORE OTHER VARIABLES CHANGE

(4) Using curvilinear regressions

The discussion of multiple correlation to this point has been limited to linear relationships—relations where the change in the dependent variable accompanying changes in each independent variable was assumed to be of exactly the same amount, no matter how large or how small the independent variable became. Thus in the farm income example, it was assumed that each additional cow would be accompanied by the same increase in income, no matter whether it was the first, the tenth or the thirtieth. Similarly, each additional acre in crops or each additional man employed was assumed to be accompanied by an identical contribution to the income, no matter how large or how small the business already was. It is quite evident that such an analysis makes no provision for there being an optimum size of operation for given circumstances, or for differences in the contributions of different numbers of units. In this particular case, it assumes that there is no such thing as the principle of diminishing returns. Such an analysis might therefore fail entirely to reveal the proper size of productive unit, or the number of each of the several elements to be employed to yield maximum returns.

In many other types of problems for which multiple correlation analysis might be used, limitation of the analysis to linear relations would seriously restrict its value or prevent its use altogether. In dealing with the effect of weather upon crop yields, several variable weather factors are usually concerned. There may be an optimum point for growth, with respect to both temperature and precipitation, with values either above or below the optimum tending to produce lower yields. Linear regressions are obviously unfitted to express such relations. In problems such as these, and many others which might be enumerated, determination of the exact curvilinear relation between independent and dependent variable, while simultaneously eliminating the effect of other factors which also affect the dependent variable, is the most important feature in the investigation. Unless the curve itself can be determined, the other conclusions are of little value.

The problem in its simplest outlines may be stated as follows: Given a series of paired observations of the values of a dependent variable X_1, and two or more independent variables X_2, X_3, X_4, etc., required to find the change in X_1 accompanying changes in X_2, X_3, and X_4, in turn, while holding the remaining independent factors constant, so that for any given values of X_2, X_3, and X_4, etc., values may be estimated for X_1, according to the regression equation

$$X_1 = a' + f(X_2) + f(X_3) + f(X_4) + \text{etc.} \tag{54}$$

The expression "$f(X_2)$" is used here simply as a perfectly general term meaning any regular change in X_1 with given changes in X_2, whether describable by a straight line or a curve. The equation is read "X_1 is a *function* of X_2 plus a *function* of X_3," etc.

The several partial (or "net") regression curves may be determined either by the use of definite mathematical expressions, one for each independent variable, with the constants all determined simultaneously just as in linear multiple correlation; or by a method known as "successive graphic approximation," which involves no prior assumptions as to the shapes of the curves.

Multiple Regression Curves Mathematically Determined.—In using definite mathematical functions, it is necessary to express the curvilinear relations by simple mathematical curves of some type, so that the constants for the curves may be determined by methods similar to those already presented. If simple parabolas were used, involving only the first and second powers of each independent variable, equation (54) could be expressed

$$X_1 = a + b_2X_2 + b_{2'}(X_2^2) + b_3X_3 + b_{3'}(X_3^2) + b_4X_4 + b_{4'}(X_4^2) \tag{55}$$

However, this type of parabola is not very flexible, and in practice it fits but very few actual curves. If the more flexible cubic parabola were employed, involving the first, second, and third powers of each independent variable, the equation would be

$$\begin{aligned} X_1 = a + b_2(X_2) + b_{2'}(X_2^2) + b_{2''}(X_2^3) + b_3(X_3) + b_{3'}(X_3^2) \\ + b_{3''}(X_3^3) + b_4(X_4) + b_{4'}(X_4^2) + b_{4''}(X_4^3) \end{aligned} \tag{56}$$

This last equation for three independent variables involves 10 constants, and increases the error in their determination accordingly, and the clerical labor of dealing with the squared and cubed values would be large (unless they were coded). Even then, it offers no guarantee that the curves for each function would truly represent the real relationship. The curves corresponding to the three functions in equation (54) would be as follows:

$$f(X_2) = b_2X_2 + b_{2'}(X_2^2) + b_{2''}(X_2^3)$$
$$f(X_3) = b_3X_3 + b_{3'}(X_3^2) + b_{3''}(X_3^3)$$
$$f(X_4) = b_4X_4 + b_{4'}(X_4^2) + b_{4''}(X_4^3)$$

Whether or not these curves would actually be a good fit to the true functions could not be told beforehand, for the problem is not to find the curves expressing the relation between X_1 and each of the other variables according to the apparent relation, but according to the underlying relation which may become apparent only when the differences in X_1 associated with differences in the other factors have been eliminated. Each of the independent factors may be correlated with the other independent factors to a greater or less degree. Thus in the problem which follows, correlating X_2 with X_3, $r = +0.07$; X_2 with X_4, 0.00; and X_3 with X_4, -0.67. The last correlation is sufficient to tend to obscure the relations. When we make a dot-chart showing the apparent relation between X_1 and X_3, we cannot tell how much of the observed differences in X_1 are due to the differences in X_4 associated with the differences in X_3. For that reason we cannot be sure what type of curve would truly represent the differences in X_1 with differences in X_3 after allowances had been made for these other factors. Even though the apparent relation might indicate that a straight line or some type of parabola would fit, there would be no guarantee that this would truly represent the net functional relationship. The successive approximation method, which makes no rigid assumption as to the type of curve, is therefore to be preferred.[1]

Multiple Regression Curves by Successive Approximations.—The general method of determining partial regression curves by the successive approximation method may be outlined as follows: The linear partial regressions are first obtained. Then the dependent variable is adjusted for the deviations from the mean of all independent variables except one, and a correlation chart, or dot-chart, is constructed between these adjusted values and that independent variable. This provides the basis for drawing in the first approximation curve for the net regression of the dependent variable on that independent variable. The dependent variable is then corrected for all except the next independent variable, the corrected values plotted against the

[1] The determination of multiple regression curves by fitting definite mathematical equations is dealt with at more length in Chapter 21.

values of that variable, and the first approximation curve determined with respect to that variable. This process is carried out for each independent variable in turn, yielding a complete set of first approximations to the net regression curves. These curves are then used as a basis for correcting the dependent factor for the approximate curvilinear effect of all independent variables except one, leaving out each in turn; and second approximation curves are determined by plotting these corrected values against the values of each independent variable in turn. New corrections are made from these curves, and the process is continued until no further change in the several regression curves is indicated.

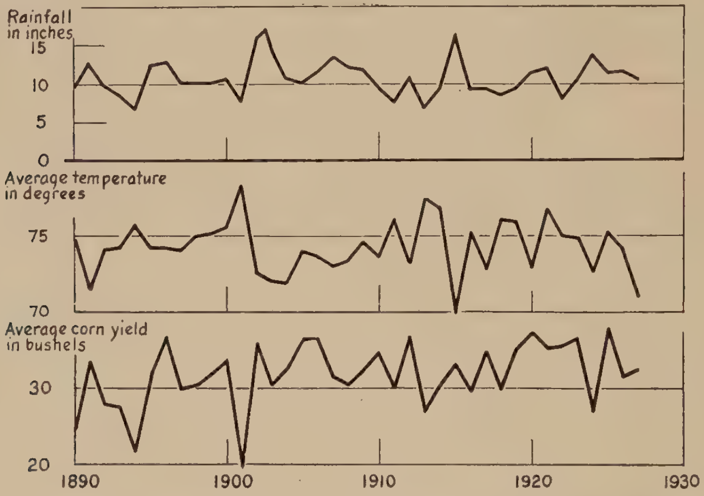

FIG. 32.—Rainfall, temperature, and corn yields in the Corn Belt, 1890 to 1927.

The process of determining net curvilinear regressions by the successive graphic approximation method may be illustrated by the data shown in Table 50. These data show, for a period of 38 years, the average rainfall during June, July, and August, for nine weather stations scattered through the Corn Belt. This precipitation has been designated as variable X_3. The average temperature during the same months, at the same stations, has been designated as X_4. The average yield of corn per acre, in the six leading Corn Belt states, is shown as X_1—the variable whose fluctuations are to be explained, so far as possible, by the other factors.

It is evident from the table that there has been a marked upward trend in corn yield during this period, while there has not been a similar trend in rainfall or temperature. Plotting each one of the three factors,

X_3, X_4, and X_1, as shown in Figure 32, we notice, however, that there have been marked long-time cycles in rainfall and temperature during the period. To a certain extent the upward swing in yields has agreed with the high point of the rainfall cycles, particularly from 1919 to 1921. It is not safe, therefore, to fit a long-time trend to yield, and to assume that in removing that trend we are merely taking out the effects of such factors as better varieties, improved methods of tillage, or concentration of acreage in the more fertile sections. Since there is some association between rainfall and time, at least over considerable periods, in eliminating all of the variation associated with time we might be eliminating a part of the variation which really reflected differences in rainfall. Accordingly we may make time itself one of the factors in the multiple correlation, and only ascribe to time that part of the long-time change in yields which is not associated with differences in rainfall or in temperature. Each year, numbered from 0 up, is therefore included as one of the factors in the multiple correlation, and is designated as variable X_2.[2]

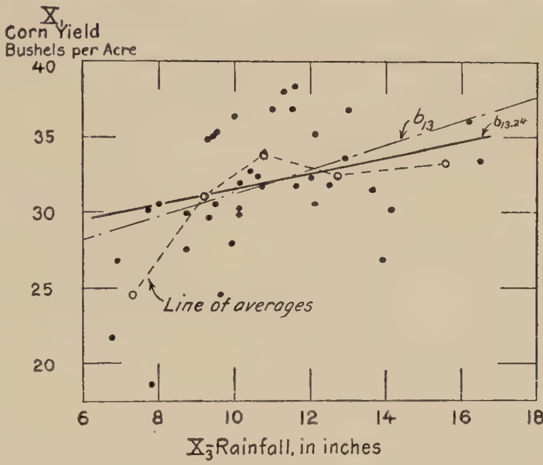

FIG. 33.—Apparent relation of corn yields to rainfall (with simple and net regression lines).

The apparent relation between rainfall (X_3) and yield (X_1) is indicated in Figure 33, by a dot-chart of the relation, with the average yield indicated for each group of years of similar rainfall. The broken line connecting these averages indicates that there is a marked curvilinear relation, the lower increases in rainfall being accompanied by much greater increases in yield than the higher increases. Fitting a straight regression line to these two variables, the relation is found to be

$$X_1 = 23.55 + 0.776X_3$$

This line is accordingly drawn in on the chart, cutting across the curve indicated by the line of group averages.

Although Figure 33 shows yields to be definitely associated with differences in rainfall, it must be noted that rainfall is significantly

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[2] Note the parallel treatment of changes in time as an independent factor in R. A. Fisher, Statistical Methods for Research Workers, 2nd ed., p. 174.

TABLE 50

YIELD OF CORN, RAINFALL, AND TEMPERATURE IN SIX LEADING STATES; AND YIELD ESTIMATED BY LINEAR REGRESSIONS ON THREE FACTORS *

Year	Time, X_2	Rainfall, in inches, X_3	Temperature, in degrees, X_4	Yield, in bushels, X_1	Estimated yield, X'_1	Difference, $X_1 - X'_1$ z
1890	0	9.6	74.8	24.5	28.4	−3.9
1891	1	12.9	71.5	33.7	31.6	2.1
1892	2	9.9	74.2	27.9	29.1	−1.2
1893	3	8.7	74.3	27.5	28.5	−1.0
1894	4	6.8	75.8	21.7	27.0	−5.3
1895	5	12.5	74.1	31.9	30.9	1.0
1896	6	13.0	74.1	36.8	31.4	5.4
1897	7	10.1	74.0	29.9	30.0	−0.1
1898	8	10.1	75.0	30.2	29.7	0.5
1899	9	10.1	75.2	32.0	29.8	2.2
1900	10	10.8	75.7	34.0	30.1	3.9
1901	11	7.8	78.4	19.4	27.5	−8.1
1902	12	16.2	72.6	36.0	34.6	1.4
1903	13	14.1	72.0	30.2	33.8	−3.6
1904	14	10.6	71.9	32.4	32.1	0.3
1905	15	10.0	74.0	36.4	31.1	5.3
1906	16	11.5	73.7	36.9	32.2	4.7
1907	17	13.6	73.0	31.5	33.7	−2.2
1908	18	12.1	73.3	30.5	32.9	−2.4
1909	19	12.0	74.6	32.3	32.5	−0.2
1910	20	9.3	73.6	34.9	31.6	3.3
1911	21	7.7	76.2	30.1	29.8	0.3
1912	22	11.0	73.2	36.9	33.0	3.9
1913	23	6.9	77.6	26.8	29.1	−2.3
1914	24	9.5	76.9	30.5	31.0	−0.5
1915	25	16.5	69.9	33.3	37.7	−4.4
1916	26	9.3	75.3	29.7	31.8	−2.1
1917	27	9.4	72.8	35.0	33.0	2.0
1918	28	8.7	76.2	29.9	31.4	−1.5
1919	29	9.5	76.0	35.2	32.1	3.1
1920	30	11.6	72.9	38.3	34.6	3.7
1921	31	12.1	76.9	35.2	33.4	1.8
1922	32	8.0	75.0	35.5	32.1	3.4
1923	33	10.7	74.8	36.7	33.8	2.9
1924	34	13.9	72.6	26.8	36.5	−9.7
1925	35	11.3	75.3	38.0	34.2	3.8
1926	36	11.6	74.1	31.7	35.0	−3.3
1927	37	10.4	71.0	32.6	35.7	−3.1

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* Data from E. G. Misner, Studies of the Relation of Weather to the Production and Price of Farm Products, I. Corn. Mimeographed publication, Cornell University, March, 1928. The six states are Iowa, Illinois, Nebraska, Missouri, Indiana, and Ohio.

correlated with X_4, temperature, the correlation being $r_{34} = -0.67$; and is also slightly correlated with time. To some extent, then, the changes in yield shown in the figure to be associated with differences in rainfall may really be due to concomitant differences in the other two factors. The extent to which these other two factors may have influenced the relations can be judged by determining the multiple correlation of X_1 with all three factors, and then noting how the regression of X_1 on X_3 alone (b_{13}), which has just been shown plotted in the figure, compares with the net regression of X_1 on X_3 ($b_{13.24}$) determined while simultaneously holding constant the linear effects of X_2 and X_4. The next step, therefore, is to determine the multiple regression equation and the coefficient of multiple correlation, according to the methods outlined in Chapters 12 and 13.

The regression equation works out to be

$$X_1 = 53.505 + 0.146X_2 + 0.537X_3 - 0.405X_4$$

and the multiple correlation, adjusted for the number of observations and constants, $\bar{R}_{1.234}$, is 0.49.[3]

This result shows that when the net linear influence of trend and of temperature is allowed for, yield increases on the average only 0.54 bushel for each increase of one inch in rainfall, whereas before these other factors were taken into account, yield appeared to increase 0.78 bushel with each additional inch of rainfall. The difference between the simple regression and the net regression may be shown by plotting the latter as well in Figure 33.[4] It is then quite apparent how different are the relations as shown by the two lines.

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[3] Using units of years of time, inches of rainfall in tenths, and degrees of temperature in tenths, and corn yields in tenth bushels, the normal equations for the data of Table 50 are as follows:

$$\begin{aligned} 4569.50b_{12.34} + 248.00b_{13.24} - 8.50b_{14.23} &= 6{,}813.00 \\ 248.00b_{12.34} + 18{,}989.06b_{13.24} - 10{,}279.41b_{14.23} &= 14{,}726.97 \\ -8.50b_{12.34} - 10{,}279.41b_{13.24} + 12{,}408.86b_{14.23} &= -8{,}442.64 \end{aligned}$$

$n\sigma_1^2 = 70{,}455.03$; $\sigma_1 = 43.0$; or 4.3 bushels.

[4] The net regression line showing the change in yield with changes in rainfall while holding constant time and temperature, may be computed from the multiple regression equation by substituting the average values for time and for temperature for X_2 and X_4, and then working out the new constant. For the data given in Table 50, the averages are as follows:

$$M_2 = 18.500;\ M_3 = 10.784;\ M_4 = 74.276;\ M_1 = 31.916$$

Substituting the means of X_2 and X_4 for their values in the multiple regression equation, that equation becomes:

$$X_1 = 53.505 + (0.146)(18.500) + 0.537X_3 - (0.405)(74.276) = 26.124 + 0.537X_3$$

The net regression line in Figure 33 is therefore drawn in from this last equation.

Considering the effect of the other factors reduces the linear regression of X_1 on X_3 by nearly $\frac{1}{3}$. If other factors have that much effect on the average linear relation, they may have an even greater effect on the shape of the curve. The net regression line in Figure 33 shows the average change in the values of X_1 with different values of X_3, after taking into account the differences in X_2 and X_4. The average yield for different groups according to rainfall, connected by the broken line, shows definitely that the simple regression line is but a poor indication of the underlying relation between X_1 and X_3. The net (or partial) regression line may be an equally poor indication of the relation with the other factors held constant. What is needed is some way of seeing the differences in the *individual* values of X_1 for different values of X_3, after the variation due to X_2 and X_4 has been eliminated. It is impossible to do this entirely, for we have as yet no measure of the *curvilinear* relation of X_1 to X_2 or X_3. But we do have our net regression coefficients, which measure the linear regression of X_1 on these other factors, and by using them we can eliminate from X_1 that part of its variation associated with the linear effects of X_2 and X_4, and then see if that gives us any clearer picture of the curvilinear relation between X_1 and X_3.

Determining the "first approximation" net regression curves.—Having determined the linear multiple regression equation, the estimated value of X_1 is next calculated for each one of the 38 observations, by substituting the corresponding values of X_2, X_3, and X_4 in the equation. Each of the estimated values (X'_1) is then subtracted from the actual value (X_1) giving the residual values (z), as also shown in Table 50.

The next step is to construct a scatter diagram to show the relation between variations in X_3, and the variation in X_1 after that associated with X_2 and X_4 has been eliminated. To do that, the net regression line for X_1 on X_3 is plotted on Figure 34, just as it had been on Figure 33.[5]

The residuals for each observation, from Table 50, are then plotted on the chart, with their X_3 value for abscissa, and with the value of z as ordinate *from the net regression line as zero base.* For the first observation, $X_3 = 9.6$ and $z = -3.9$. The ordinate of the point on the net

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[5] To plot the line, all that is necessary is to take the equation of the line to be used (see previous footnote)

$$X_1 = 26.124 + 0.537X_3$$

and substitute any two convenient values for X_3, say 6 and 16.

$$\text{For } X_3 = 6,\ X_1 = 26.124 + (0.537)(6) = 29.35$$
$$\text{For } X_3 = 16,\ X_1 = 26.124 + (0.537)(16) = 34.71$$

With these two sets of coordinates, the line is then drawn in with a straight edge through the points indicated.

regression line corresponding to $X_3 = 9.6$ is 31.3, and the dot for this observation is correspondingly plotted 3.9 lower than that, at 27.4. For the second observation, $X_3 = 12.9$ and $z = + 2.1$. The ordinate of the point on the regression line corresponding to $X_3 = 12.9$ is 33.1; so the dot for this observation is plotted at 33.1 + 2.1, or 35.2. After the corresponding operation has been carried out for all the observations, the figure appears as shown in Figure 34.[6]

If Figure 34 is compared with Figure 33, it is readily seen that the scatter of the dots has been reduced. This will always be true when the other variables show any significant relation to the dependent factor; that is, when $\bar{R}_{1.234}$ exceeds $\bar{r}_{13}$. The scatter is reduced because that part of the variation in X_1 which can be expressed as net linear functions of X_2 and X_4 has now been eliminated.[7]

Consideration of Figure 34 can be facilitated by computing the means of the ordinates corresponding to each value of X_3. These can be obtained by simply averaging together the z values for each value of X_3, and plotting those averages as deviations from the regression line, just as the individual deviations were plotted previously. The necessary averages are as shown in table 51 on p. 196.

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[6] The simplest way of plotting the individual observations is to use a scale, which can be slid along the regression line as zero. The values of z are then plotted directly as vertical deviations from the points on the regression line corresponding to the particular values of the independent variable considered, as X_3 in the present case.

[7] This can be readily proved. Each point on the net regression line was obtained by the formula:

$$\text{(A)} \quad X_1 = a_{1.234} + b_{12.34}M_2 + b_{13.24}X_3 + b_{14.23}M_4$$

To these values have been added the residuals, z. These residuals equal $X_1 - X'_1$, and therefore for each observation are equal to

$$\text{(B)} \quad X_1 - a_{1.234} - b_{12.34}X_2 - b_{13.24}X_3 - b_{14.23}X_4$$

The ordinate of each dot in Figure 34 is the ordinate of the regression line plus z, and is therefore equal to the sum of the two equations, (A) and (B). If we use π to represent these ordinates, they are therefore equal to

$$\pi = a_{1.234} + b_{12.34}M_2 + b_{13.24}X_3 + b_{14.23}M_4 + X_1 - a_{1.234} - b_{12.34}X_2$$
$$- b_{13.24}X_3 - b_{14.23}X_4$$

$$\pi = X_1 - b_{12.34}(X_2 - M_2) - b_{14.23}(X_4 - M_4)$$

$$\pi = X_1 - b_{12.34}x_2 - b_{14.23}x_4$$

The adjusted values shown on Figure 34 are therefore simply the values of X_1, less net linear corrections for deviations in X_2 and X_4 from their mean values.

TABLE 51

AVERAGE VALUES OF z, FOR CORRESPONDING X_3 VALUES

X_3 values	Number of cases	Average of X_3	Average of z
Under 8.0	4	7.30	−3.85
8.0– 9.9	10	9.19	+0.16
10.0–10.9	8	10.35	+1.49
11.0–11.9	5	11.40	+2.56
12.0–13.9	8	12.76	−0.52
14.0 and over	3	15.60	−2.20

These averages, when plotted the same as the individual observations, and connected by a broken line, give the irregular line also shown in Figure 34. Comparing this line with the similar one in Figure 33, it is seen that, though the lines are in general similar, there are some marked differences. The average for the second group (X_3 = 8.0–9.9) is now above the straight net regression line, while previously it was below it. Likewise the average for X_3 = 14 and over is now slightly below the average for X_3 = 12.0 to 13.9, while before it was a little above it. Also, the difference between the first two averages is not so large as it appeared before. Apparently part of the previous deviations reflected other independent factors.

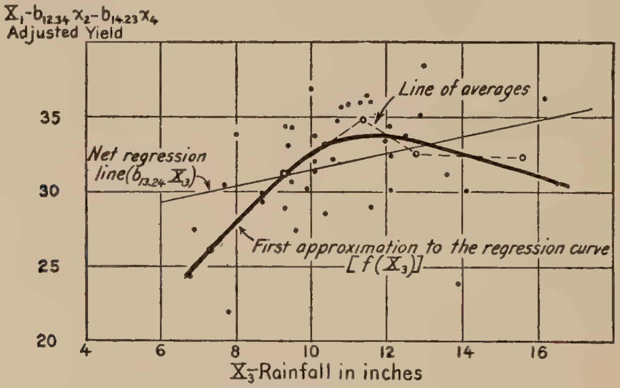

FIG. 34.—Rainfall and yield of corn adjusted to average temperature and year, and first approximation curve fitted to the averages.

It is quite evident that a regression curve is indicated, rising sharply to a maximum yield between 10 and 12 inches of rain, then declining gradually for higher rainfalls. Such a curve is accordingly drawn in free-hand, passing as near to the several group averages as is consistent with a continuous smooth curve. This curve is the first approximation to the curvilinear function

$$X_1 = f(X_3)$$

which was required to be determined while simultaneously taking into effect the curvilinear effects of X_2 and X_4 on X_1. It is only a first approximation because it has been determined while allowing for only the net *linear* effects of the other two variables. If their *curvilinear* effect were determined and allowed for, that might change somewhat the shape of this curve.

The next step is to determine similar first approximations to the curvilinear relation between X_1 and X_2, and between X_1 and X_4, with the net linear effects of the other variables eliminated just as has been done for X_3. It is not necessary to plot the apparent relation between X_1 and X_2 or X_1 and X_4. This was done in the case of X_3 (Figure 33) solely to illustrate the difference between taking the apparent relations, or taking the net relations after the linear influence of the other factors had been allowed for (Figure 34). Instead, we may proceed at once to determine the net relations for X_1 to X_2. Figure 35 shows this step. This figure is constructed exactly as was Figure 34, by the following steps: (1) Plot the net regression line.[8] (2) Plot in the individual

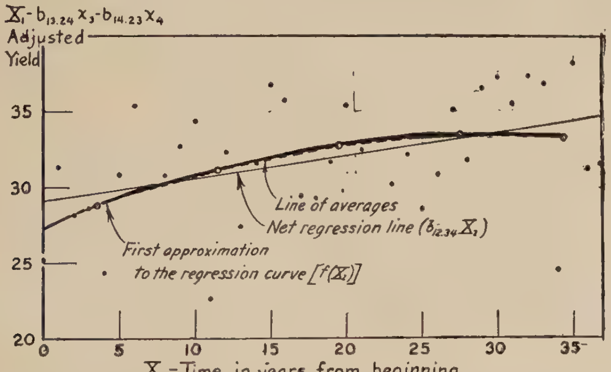

FIG. 35.—Time and yield of corn adjusted to average temperature and rainfall, and first approximation curve fitted to the averages.

[8] The regression equation, for mean values of X_3 and X_4, becomes

$$\begin{aligned} X_1 &= 53.505 + 0.146X_2 + 0.537(M_3) - 0.405(M_4) \\ &= 53.505 + 0.146X_2 + (0.537)(10.784) - (0.405)(74.276) \\ &= 29.214 + 0.146X_2 \end{aligned}$$

This equation is then the equation to which the net regression line in Figure 35 is drawn. Substituting the values $X_2 = 0$ and $X_2 = 20$ in the equation, values for X_1 of 29.214 and 32.13 are obtained, giving the coordinate points for drawing in the line.

residuals, z, as deviations from that line.[9] (3) Average the residuals grouped according to X_2, plot the group averages and connect them by a broken line. (4) Draw in a smooth curve through the line of averages, if a curve is indicated.

After the first two steps have been carried out, just as described for Figure 34, grouping and averaging the residuals with respect to X_2 gives the averages shown in Table 52.

TABLE 52

AVERAGE VALUES OF z FOR CORRESPONDING X_2 VALUES

X_2 values	Number of cases	Average of X_2	Average of z
0– 7	8	3.5	−0.38
8–15	8	11.5	+0.24
16–23	8	19.5	+0.64
24–31	8	27.5	+0.26
32–37	6	34.5	−1.00

The average residuals shown in the table are then plotted in above and below the regression line in Figure 35, and connected by a broken line. This line of averages indicates that corn yield (for years of similar rainfall and temperature) rose rapidly during the earlier years, then more and more gradually, until during the last ten years it tended to remain about on the same level. A smooth continuous curve is therefore drawn through the averages, completing step (4), and giving the first approximation to the curvilinear net regression of X_1 on X_2, $f(X_2)$.

The same operations are then carried out for X_4, as shown in Figure 36. After drawing in the net regression line,[10] and plotting in the individual observations,[11] the residuals are grouped on X_4 and averaged, with the results shown in Table 53.

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[9] For the first observation, $X_2 = 0$, and $z = -3.9$. The point on the regression line corresponding to $X_2 = 0$ has an ordinate of 29.2. The dot for this observation is accordingly plotted at $29.2 - 3.9$, or 25.3. For the next observation, $X_2 = 1$, and $z = 2.1$. The corresponding ordinate on the regression line is 29.4, so the dot is plotted at $29.4 + 2.1$, or 31.5. The dot for each observation is plotted in turn in the same way, using a sliding graphic scale to place the dots above or below the regression line.

[10] The net regression line for X_1 and X_4 may be determined by an alternative method to that used before. On such charts as Figures 34, 35 or 36, the net regression line will always pass through the mean of the two variables. For Figure 36 therefore, X_1 will have its mean value, 31.92, when X_4 has its mean value, 74.28. From the net regression coefficient, $b_{14 \cdot 23}$, it is evident that each unit increase in X_4 is accompanied by -0.405 units increase in X_1. If X_4 is increased from 74.28 to 78.28, or 4 units, X_1 will change by $(-0.405)(4)$, or -1.62. For $X_4 = 78.28$, X_1

TABLE 53

AVERAGE VALUES OF z FOR CORRESPONDING X_4 VALUES

X_4 values	Number of cases	Average of X_4	Average of z
Under 72.0	4	71.08	−1.28
72.0–72.9	5	72.58	−1.24
73.0–73.9	5	73.36	+1.46
74.0–74.9	10	74.30	+0.49
75.0–75.9	7	75.33	+0.91
76.0–76.9	5	76.44	+0.64
77.0 and over	2	78.00	−5.20
76.0 and over	7	76.89	−1.03

The last group, on the first grouping, has but two cases, so the last two groups are combined, giving the averages shown in the last line. The fact that both the items above 77 degrees are low, also evident in Figure 36, would give a little more reliability to the average based on only two items; but it is generally best not to give such an extreme bend to the end of a regression curve as this would call for, at least on the first approximation. The larger grouping will therefore be used in this case, leaving the subsequent work to determine whether the more extreme bend is justified.

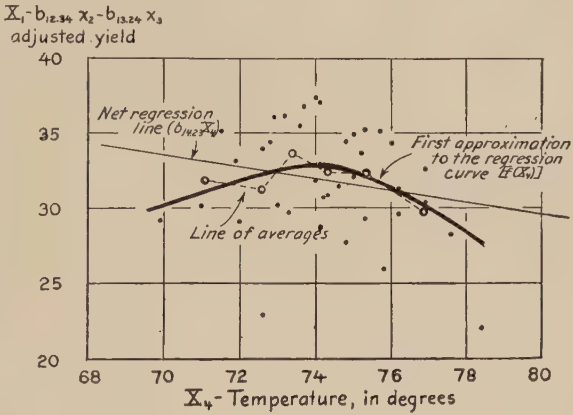

FIG. 36.—Temperature and yield of corn adjusted to average rainfall and year, and first approximation curve fitted to the averages.

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will therefore be 31.92 − 1.62, or 30.30. This gives the two sets of points necessary to locate the line; when $X_4 = 74.28$, $X_1 = 31.92$; and when $X_4 = 78.28$, $X_1 = 30.30$.

[11] The individual residuals are plotted in the same way as indicated in the other two cases; the residual −3.9 for $X_4 = 74.8$ is plotted 3.9 units below the corresponding point on the regression line, and similarly for the other observations.

The line of averages in Figure 36 indicates that yields may tend to rise as temperature increases up to between 73 and 75 degrees, and then to fall as the temperature goes still higher. A smooth curve is therefore drawn in, averaging out the irregularities shown in the broken line of the group averages. It does not make much difference if these first approximation curves are not drawn in in exactly the right position or shape, as the subsequent operations will tend to correct them to the proper shape if the original one is incorrect. It is for that reason that fairly accurate results can be secured by this graphic process, even though the true shape of the curves is not known at the beginning.

Estimating X_1 from the first approximation curves.—We have now arrived at first approximations to the net regression curves for X_1, against each of the three factors. It must be remembered that in making the adjustments on X_1 to arrive at these curves, only the net *linear* effects of the other independent variables have been eliminated. Now that we have at least an approximate measure of the curvilinear relations of X_1 to the independent variables, making adjustments to eliminate these approximate curvilinear effects may enable us to determine more accurately the true curvilinear relation to each variable.

The first step in the next stage of the process is to work out estimated values of X_1 based on the curvilinear relations. To do this we may designate the relation between X_1 and X_2 shown by the curve in Figure 35 as $f'(X_2)$; the relation between X_1 and X_3 shown in Figure 34 as $f'(X_3)$; and the relation between X_1 and X_4 shown in Figure 36 as $f'(X_4)$. The estimates of X_1 may then be worked out by the regression equation

$$X''_1 = a'_{1.234} + f'(X_2) + f'(X_3) + f'(X_4) \qquad (57)$$

The symbol X''_1 is used to designate this second set of estimates, just as X'_1 was used to designate the first set, worked out from the linear regression equation. The constant $a'_{1\cdot234}$ is different from the constant $a_{12\cdot34}$ used in equation (36); its value is given by the formula

$$a'_{1.234} = M_1 - \frac{\Sigma[f'(X_2) + f'(X_3) + f'(X_4)]}{n} \qquad (58)$$

To work out $a'_{12\cdot34}$ according to equation (58), it is first necessary to work out the value $f'(X_2) + f'(X_3) + f'(X_4)$ for each set of observations. For the first observation, for example, $X_2 = 0$, $X_3 = 9.6$, and $X_4 = 74.8$. From $f'(X_2)$, given in Figure 35, the curve reading (or ordinate) corresponding to a value of 0 for X_2 is 27.3. For $f'(X_3)$, Figure 34, the ordinate of the curve corresponding to $X_3 = 9.6$ is 31.7. For $f'(X_4)$, Figure 36, the curve ordinate corresponding to $X_4 = 74.8$ is 32.5. The value $[f'(X_2) + f'(X_3) + f'(X_4)]$ for the first observation is therefore $[27.3 + 31.7 + 32.5]$, or 91.5. The sum of these values for each observation is the value required in equation (58).

Before continuing the process of reading each value from the charts for the remaining observations, it should be noted that since many observations of each variable have the same values, the same point would be read from each chart many times. The process of working out the computations can be much simplified by reading each required value from each chart once for all, and recording it so that it can be used each time. Since each chart indicates each individual observation for each independent variable, only those points for which there are observations need be recorded. Carrying out this process, the functional relations may be recorded as shown in Tables 54, 55, and 56, which show the readings from Figures 35, 34, and 36, respectively.[12]

TABLE 54

Values of X_1 Corresponding to Given Values of X_2, from the First Approximation Curve

X_2	$f'(X_2)$	X_2	$f'(X_2)$	X_2	$f'(X_2)$	X_2	$f'(X_2)$
0	27.3	10	30.8	20	32.8	29	33.4
1	27.8	11	31.0	21	33.0	30	33.5
2	28.2	12	31.3	22	33.1	31	33.5
3	28.6	13	31.5	23	33.1	32	33.5
4	29.0	14	31.7	24	33.2	33	33.5
5	29.4	15	31.9	25	33.2	34	33.5
6	29.7	16	32.1	26	33.3	35	33.5
7	30.0	17	32.3	27	33.3	36	33.5
8	30.3	18	32.5	28	33.4	37	33.5
9	30.6	19	32.6				

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[12] In entering these values it is not worth while reading further than the first decimal, for the line is not drawn more accurately than to within 0.1 or 0.2. The accuracy depends, of course, on the scale; but it is not worth using very large charts to secure spuriously high accuracy, when the standard error of any particular point on the curve is probably several units, and when the curve is only a first approximation, subject to subsequent modification.

TABLE 55

Values of X_1 Corresponding to Given Values of X_3, from the First Approximation Curve

X_3	$f'(X_3)$	X_3	$f'(X_3)$	X_3	$f'(X_3)$	X_3	$f'(X_3)$
6.8	24.6	9.5	31.5	10.8	33.4	12.9	33.3
6.9	25.0	9.6	31.7	11.0	33.5	13.0	33.2
7.7	27.1	9.9	32.4	11.3	33.6	13.6	32.9
7.8	27.4	10.0	32.5	11.5	33.7	13.9	32.7
8.0	27.9	10.1	32.6	11.6	33.7	14.1	32.5
8.7	29.7	10.4	33.1	12.0	33.7	16.2	31.0
9.3	31.0	10.6	33.3	12.1	33.6	16.5	30.8
9.4	31.2	10.7	33.4	12.5	33.5		

TABLE 56

Values of X_1 Corresponding to Given Values of X_4, from the First Approximation Curve

X_4	$f'(X_4)$	X_4	$f'(X_4)$	X_4	$f'(X_4)$	X_4	$f'(X_4)$
69.9	30.2	73.0	32.5	74.2	32.8	75.7	31.6
71.0	31.0	73.2	32.6	74.3	32.7	75.8	31.5
71.5	31.4	73.3	32.6	74.6	32.6	76.0	31.3
71.9	31.7	73.6	32.7	74.8	32.5	76.2	31.0
72.0	31.8	73.7	32.7	75.0	32.3	76.9	30.1
72.6	32.2	74.0	32.8	75.2	32.1	77.6	29.0
72.8	32.3	74.1	32.8	75.3	32.0	78.4	27.6
72.9	32.4						

The values to determine $a'_{1.234}$ may now be worked out in orderly manner, as shown in Table 57, in the fourth to the seventh columns.

This computation gives us the sum of the respective functional values for the 38 observations. Substituting this sum and the number of observations in equation (58), we find the required constant to be

$$a'_{1.234} = 31.916 - \frac{3621.9}{38} = -63.397$$

Since the functional values for our regression equation are only expressed to one decimal point, we will use -63.4 for $a'_{1.234}$, which will result in the estimated values being 0.003 units too low, on the average.

It is now possible to complete the process of computing X''_1, the estimated value of X_1, using the first approximation curves, according

TABLE 57

Computation of Functional Values Corresponding to Independent Variables, of the Estimated Value of X_1, and the New Residual, for Each Observation

X_2	X_3	X_4	$f'(X_2)$	$f'(X_3)$	$f'(X_4)$	$f'(X_2)$ $+f'(X_3)$ $+f'(X_4)$	$\Sigma(f)+a'$ $=X_1''$	X_1	X_1-X_1'' z''
0	9.6	74.8	27.3	31.7	32.5	91.5	28.1	24.5	−3.6
1	12.9	71.5	27.8	33.3	31.4	92.5	29.1	33.7	4.6
2	9.9	74.2	28.2	32.4	32.8	93.4	30.0	27.9	−2.1
3	8.7	74.3	28.6	29.7	32.7	91.0	27.6	27.5	−0.1
4	6.8	75.8	29.0	24.6	31.5	85.1	21.7	21.7	0
5	12.5	74.1	29.4	33.5	32.8	95.7	32.3	31.9	−0.4
6	13.0	74.1	29.7	33.2	32.8	95.7	32.3	36.8	4.5
7	10.1	74.0	30.0	32.6	32.8	95.4	32.0	29.9	−2.1
8	10.1	75.0	30.3	32.6	32.3	95.2	31.8	30.2	−1.6
9	10.1	75.2	30.6	32.6	32.1	95.3	31.9	32.0	0.1
10	10.8	75.7	30.8	33.4	31.6	95.8	32.4	34.0	1.6
11	7.8	78.4	31.0	27.4	27.6	86.0	22.6	19.4	−3.2
12	16.2	72.6	31.3	31.0	32.2	94.5	31.1	36.0	4.9
13	14.1	72.0	31.5	32.5	31.8	95.8	32.4	30.2	−2.2
14	10.6	71.9	31.7	33.3	31.7	96.7	33.3	32.4	−0.9
15	10.0	74.0	31.9	32.5	32.8	97.2	33.8	36.4	2.6
16	11.5	73.7	32.1	33.7	32.7	98.5	35.1	36.9	1.8
17	13.6	73.0	32.3	32.9	32.5	97.7	34.3	31.5	−2.8
18	12.1	73.3	32.5	33.6	32.6	98.7	35.3	30.5	−4.8
19	12.0	74.6	32.6	33.7	32.6	98.9	35.5	32.3	−3.2
20	9.3	73.6	32.8	31.0	32.7	96.5	33.1	34.9	1.8
21	7.7	76.2	33.0	27.1	31.0	91.1	27.7	30.1	2.4
22	11.0	73.2	33.1	33.5	32.6	99.2	35.8	36.9	1.1
23	6.9	77.6	33.1	25.0	29.0	87.1	23.7	26.8	3.1
24	9.5	76.9	33.2	31.5	30.1	94.8	31.4	30.5	−0.9
25	16.5	69.9	33.2	30.8	30.2	94.2	30.8	33.3	2.5
26	9.3	75.3	33.3	31.0	32.0	96.3	32.9	29.7	−3.2
27	9.4	72.8	33.3	31.2	32.3	96.8	33.4	35.0	1.6
28	8.7	76.2	33.4	29.7	31.0	94.1	30.7	29.9	−0.8
29	9.5	76.0	33.4	31.5	31.3	96.2	32.8	35.2	2.4
30	11.6	72.9	33.5	33.7	32.4	99.6	36.2	38.3	2.1
31	12.1	76.9	33.5	33.6	30.1	97.2	33.8	35.2	1.4
32	8.0	75.0	33.5	27.9	32.3	93.7	30.3	35.5	5.2
33	10.7	74.8	33.5	33.4	32.5	99.4	36.0	36.7	0.7
34	13.9	72.6	33.5	32.7	32.2	98.4	35.0	26.8	−8.2
35	11.3	75.3	33.5	33.6	32.0	99.1	35.7	38.0	2.3
36	11.6	74.1	33.5	33.7	32.8	100.0	36.6	31.7	−4.9
37	10.4	71.0	33.5	33.1	31.0	97.6	34.2	32.6	−1.6
Totals...	....	....	1208.4	1204.2	1209.3	3621.9			

to equation (57), using the constant which has just been computed. Comparing equations (57) and (58), it is evident that except for the constant term, X''_1 is equal to the values that have just been computed in the 7th column of Table 57. Accordingly, all that is necessary is to subtract 63.4 from each of those values. This step is also shown in Table 57, in the 8th column.

The column headed X''_1 shows the estimated values obtained by this process. The next step is to see whether the new estimates come any nearer to reproducing the observed values of X_1 than did the first set of estimates, based on the linear regression equation. We therefore compute a new set of residuals, z'', by subtracting the new estimates from the actual values of X_1. This step, also, is shown in Table 57.

$$z'' = X_1 - X''_1 \tag{59}$$

If the individual residuals shown are compared with the residuals obtained by the linear regression, as computed in Table 50, it will be seen that in general the new residuals are smaller than the previous ones, though the reverse is true in many cases. There are 23 cases in which the new residual is smaller, and 15 in which it is larger than the original residual. A more accurate comparison can be obtained by comparing the standard deviations of the residuals for the two sets. For the linear correlation, the standard deviation of the residuals was 3.6 bushels, while the standard deviations of the new residuals is 3.0 bushels. Apparently the new estimates do come nearer to the observed values, on the average, than did the first set of estimates.

Determining the second approximation regression curves.—The regression curves used in constructing the estimate X''_1 were only the first approximations to the true curvilinear relations, since they were determined by eliminating only the linear effects of the other independent factors. Now that the residuals obtained by the use of the first approximation curves have been computed, however, we can determine whether any change in the shape of the several curves is necessary.

To do this we construct Figure 37 by drawing in the regression curve from Figure 35, using the same scale as before. Use of Table 54 makes it easier to reproduce the curve. Next we plot each of the last residuals as a deviation just as before, except that now the residuals are plotted as deviations from the regression curve, instead of from the regression line, at the point corresponding to the independent variable X_2. Thus the first observation, with $X_2 = 0$, has $z'' = -3.6$. The point on the curve corresponding to $X_2 = 0$ is 27.3; so the dot has for ordinate $27.3 - 3.6$, or 23.7. The next observation is $X_2 = 1$, and $z'' = 4.6$. The corresponding value of $f'(X_2)$ is 27.8, so the ordinate for the dot is $27.8 + 4.6$, or 32.4. The coordinates for this dot are therefore 1 and 32.4. The remaining observations are plotted in the same manner, shortening the process by scaling the value for z'' directly above or below from the corresponding point on the regression curve.

With the dots all plotted, it is evident that the scatter is too great to indicate definitely changes which may be needed in the curve, if any, simply from the dots alone. Accordingly the residuals are averaged in groups, employing the same grouping as before (Table 52)

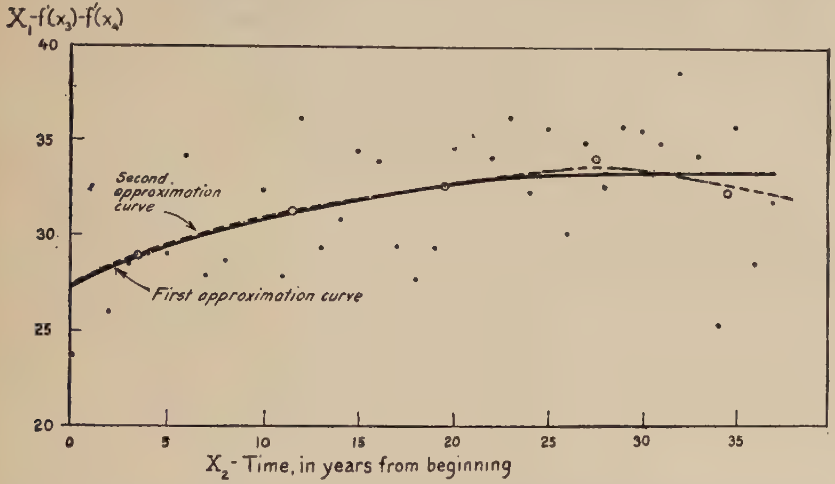

FIG. 37.—Time, and yield of corn adjusted to average temperature and rainfall on basis of first approximation curves; and second approximation to $f(X_2)$.

which eliminates the need of averaging the corresponding X_2 values over again. The new averages work out as shown in Table 58.

TABLE 58

AVERAGE VALUES OF z'', FOR CORRESPONDING X_2 VALUES

X_2 values	Number of cases	Average of X_2	Average of z''
0– 7	8	3.5	+0.10
8–15	8	11.5	+0.16
16–23	8	19.5	−0.08
24–31	8	27.5	+0.64
31–37	6	34.5	−1.08

The averages are next plotted as deviations from the first approximation curve. They indicate that a slight raise in the lower part of the curve may be needed, while a downward bend toward the end is suggested. It appears that now that the influence of rainfall and temperature on yield have been more accurately allowed for, the upward trend with time is not quite so great as it seemed before in the early years;

while the trend seems to have turned downward toward the end of the series—the exact year of the turn is indeterminate. A new curve is therefore drawn in in Figure 37, and, as it happens, a smooth, continuous curve can be drawn exactly through each of the first three group averages, but not having the extreme bend indicated by the last two group averages.

The same process may now be applied to X_3, to see if any change need be made in the first regression curve for the change in X_1 with changes in that variable. This process is carried out as shown in Figure 38, the first approximation curve being drawn in just as before, using the data given in Table 55.

Instead of plotting the individual residuals for each observation, as was just done with respect to X_2, we may proceed at once to compute the average residuals for each of the values of X_3, since it is sufficiently apparent from Figure 37 that the scatter of the individual observations is still too great to serve as a guide in correcting the first approximation curves. Averaging the residuals gives the averages shown in Table 59.

TABLE 59

AVERAGE VALUES OF z'', FOR CORRESPONDING X_3 VALUES

X_3 values	Number of cases	Average of X_3	Average of z''	Average of X_3	Average of z''
Under 8.0	4	7.30	+0.58		
8.0– 9.9	10	9.19	+0.03		
10.0–10.9	8	10.35	−0.15	10.75	+0.09
11.0–11.9	5	11.40	+0.48		
12.0–13.9	8	12.76	−1.11	13.53	−0.34
14.0 and over	3	15.60	+1.73		

Again the averages are somewhat irregular when plotted, so the last four groups are reduced to two, and then averages plotted and indicated separately. The number of observations represented by each of the first set of averages is indicated next to it, so that averages based on a small number of observations will not be given undue weight in drawing in the curve. It might be desirable in some cases, also, to try regrouping the cases into different groups—say from 8.5 to 9.4, 9.5 to 10.4, etc.—and see if that would change at all the indications as to the shifts needed in the first curve. Working that out in this case, the changes needed are still found to be about the same as shown by the group averages in Figure 38, though somewhat less regular, owing to the smaller size of groups. A new curve is then drawn in free-hand, as indicated by the group averages; rising somewhat higher than formerly at both ends, and not rising quite so high in the central portion as before.

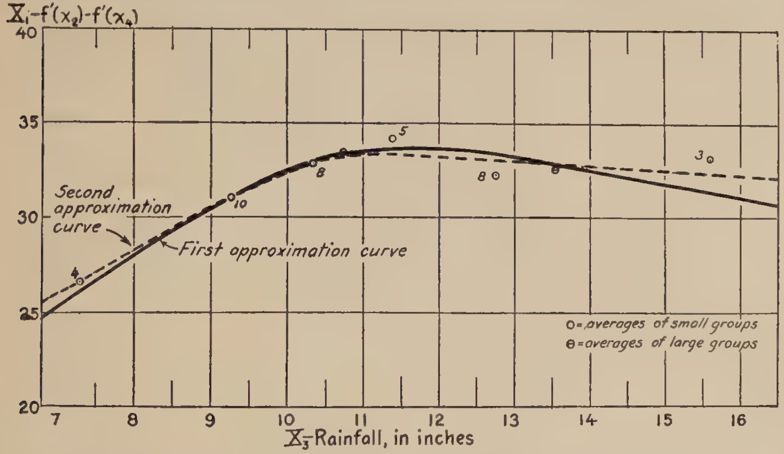

FIG. 38.—Rainfall, and yield of corn adjusted to average temperature and time on the basis of first approximation curves; and second approximation to $f(X_3)$.

Turning to the relation between X_1 and X_4, the first approximation curve for $f'(X_4)$ is reproduced in Figure 39, using the values given in

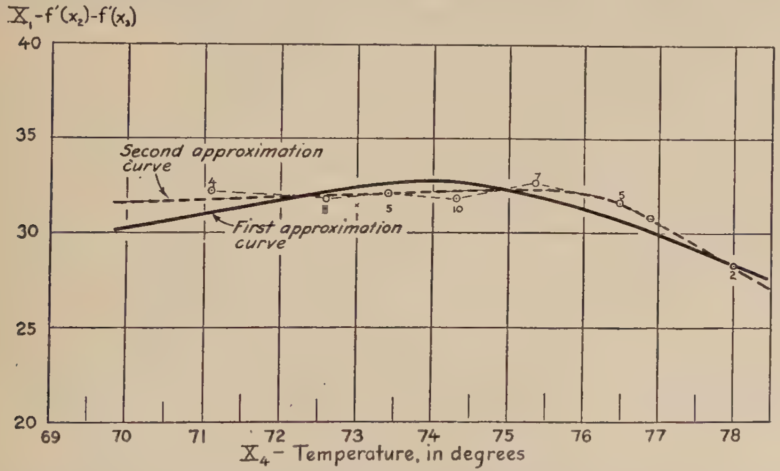

FIG. 39.—Temperature, and yield of corn adjusted to average rainfall and time on the basis of first approximation curves; and second approximation to $f(X_4)$.

Table 56. The next step is to average the values of z'' for corresponding values of X_4. Using the same groupings used in Table 53, the averages shown in the following table are arrived at.

TABLE 60

AVERAGE VALUES OF z'', FOR CORRESPONDING X_4 VALUES

X_4 values	Number of cases	Average of X_4	Average of z''
Under 72.0	4	71.08	+1.15
72.0–72.9	5	72.58	−0.36
73.0–73.9	5	73.36	−0.58
74.0–74.9	10	74.30	−0.86
75.0–75.9	7	75.33	+0.63
76.0–76.9	5	76.44	+0.90
77.0 and over	2	78.00	−0.05
76.0 and over	7	76.89	+0.63

Plotting these new averages, and connecting them by a broken line, it appears that the relation of yield to temperature may be quite different from the way it appeared on the first approximation. Apparently the highest yields are obtained around 75 to 76 degrees, instead of at 74 degrees; higher temperatures appear to reduce the yield markedly, but lower temperatures have only a slight influence on the yield. The new curve, drawn in free-hand so as to pass as nearly through these new averages as possible and still maintain a smooth continuous shape, with only a single maximum, expresses these relations.

Estimating X_1 from the second approximation curves.—Now that the second approximation curves have been determined for each variable,

TABLE 61

VALUES OF X_1 CORRESPONDING TO GIVEN VALUES OF X_2, FROM THE SECOND APPROXIMATION CURVE

X_2	$f''(X_2)$	X_2	$f''(X_2)$	X_2	$f''(X_2)$	X_2	$f''(X_2)$
0	27.4	10	31.0	20	32.7	29	33.6
1	27.9	11	31.2	21	33.0	30	33.5
2	28.4	12	31.4	22	33.2	31	33.4
3	28.8	13	31.6	23	33.3	32	33.2
4	29.2	14	31.8	24	33.4	33	33.0
5	29.5	15	32.0	25	33.5	34	32.8
6	29.8	16	32.1	26	33.6	35	32.6
7	30.2	17	32.3	27	33.7	36	32.4
8	30.4	18	32.5	28	33.7	37	32.2
9	30.7	19	32.6				

we can proceed to estimate values of X_1 on the basis of the revised curves, to see whether the new curves enable us to estimate X_1 any more accurately than the first set of curves did. To facilitate the process we first construct tables for $f''(X_2)$, $f''(X_3)$, and $f''(X_4)$, showing the readings for the functions from the revised curves.

TABLE 62

Values of X_1 Corresponding to Given Values of X_3, from the Second Approximation Curve

X_3	$f''(X_3)$	X_3	$f''(X_3)$	X_3	$f''(X_3)$	X_3	$f''(X_3)$
6.8	25.5	9.5	31.5	10.8	33.3	12.9	33.0
6.9	25.7	9.6	31.7	11.0	33.4	13.0	33.0
7.7	27.5	9.9	32.2	11.3	33.4	13.6	32.8
7.8	27.8	10.0	32.3	11.5	33.3	13.9	32.7
8.0	28.2	10.1	32.5	11.6	33.3	14.1	32.7
8.7	29.9	10.4	32.9	12.0	33.2	16.2	32.2
9.3	31.1	10.6	33.1	12.1	33.2	16.5	32.1
9.4	31.3	10.7	33.2	12.5	33.1		

TABLE 63

Values of X_1 Corresponding to Given Values of X_4, from the Second Approximation Curve

X_4	$f''(X_4)$	X_4	$f''(X_4)$	X_4	$f''(X_4)$	X_4	$f''(X_4)$
69.9	31.6	73.0	32.0	74.2	32.2	75.7	32.2
71.0	31.7	73.2	32.0	74.3	32.2	75.8	32.2
71.5	31.8	73.3	32.0	74.6	32.2	76.0	32.1
71.9	31.8	73.6	32.1	74.8	32.3	76.2	32.0
72.0	31.8	73.7	32.1	75.0	32.3	76.9	30.7
72.6	31.9	74.0	32.2	75.2	32.3	77.6	29.1
72.8	32.0	74.1	32.2	75.3	32.3	78.4	27.3
72.9	32.0						

To simplify the calculations, 20 is subtracted from each of the functional values in making subsequent entries. The computations to determine the estimated values are then carried out as shown in detail in Table 64 following, just as for Table 57. In practical computation these entries, for the second approximation curves, would be made on the same sheet as were the entries in Table 57 for the first approximation curves, thus eliminating the work of entering the values of X_1, X_2, X_3, and X_4 over again.

TABLE 64

Computation of Functional Values, from the Second Approximation Curves, Corresponding to Independent Variables for Each Observation, and Computation of Estimated Value for X_1 and of New Residuals

Independent variables			Corresponding functional values *			$f''(X_2)$ $+f''(X_3)$ $+f''(X_4)$	(7) $- a$ $= X'''_1$	Dependent variable X_1	$X_1 - X'''_1$ z'''
X_2	X_3	X_4	$f''(X_2)$	$f''(X_3)$	$f''(X_4)$				
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
0	9.6	74.8	7.4	11.7	12.3	31.4	28.0	24.5	−3.5
1	12.9	71.5	7.9	13.0	11.8	32.7	29.3	33.7	4.4
2	9.9	74.2	8.4	12.2	12.2	32.8	29.4	27.9	−1.5
3	8.7	74.3	8.8	9.9	12.2	30.9	27.5	27.5	0
4	6.8	75.8	9.2	5.5	12.2	26.9	23.5	21.7	−1.8
5	12.5	74.1	9.5	13.1	12.2	34.8	31.4	31.9	0.5
6	13.0	74.1	9.8	13.0	12.2	35.0	31.6	36.8	5.2
7	10.1	74.0	10.2	12.5	12.2	34.9	31.5	29.9	−1.6
8	10.1	75.0	10.4	12.5	12.3	35.2	31.8	30.2	−1.6
9	10.1	75.2	10.7	12.5	12.3	35.5	32.1	32.0	−0.1
10	10.8	75.7	11.0	13.3	12.2	36.5	33.1	34.0	0.9
11	7.8	78.4	11.2	7.8	7.3	26.3	22.9	19.4	−3.5
12	16.2	72.6	11.4	12.2	11.9	35.5	32.1	36.0	3.9
13	14.1	72.0	11.6	12.7	11.8	36.1	32.7	30.2	−2.5
14	10.6	71.9	11.8	13.1	11.8	36.7	33.3	32.4	−0.9
15	10.0	74.0	12.0	12.3	12.2	36.5	33.1	36.4	3.3
16	11.5	73.7	12.1	13.3	12.1	37.5	34.1	36.9	2.8
17	13.6	73.0	12.3	12.8	12.0	37.1	33.7	31.5	−2.2
18	12.1	73.3	12.5	13.2	12.0	37.7	34.3	30.5	−3.8
19	12.0	74.6	12.6	13.2	12.2	38.0	34.6	32.3	−2.3
20	9.3	73.6	12.7	11.1	12.1	35.9	32.5	34.9	2.4
21	7.7	76.2	13.0	7.5	12.0	32.5	29.1	30.1	1.0
22	11.0	73.2	13.2	13.4	12.0	38.6	35.2	36.9	1.7
23	6.9	77.6	13.3	5.7	9.1	28.1	24.7	26.8	2.1
24	9.5	76.9	13.4	11.5	10.7	35.6	32.2	30.5	−1.7
25	16.5	69.9	13.5	12.1	11.6	37.2	33.8	33.3	−0.5
26	9.3	75.3	13.6	11.1	12.3	37.0	33.6	29.7	−3.9
27	9.4	72.8	13.7	11.3	12.0	37.0	33.6	35.0	1.4
28	8.7	76.2	13.7	9.9	12.0	35.6	32.2	29.9	−2.3
29	9.5	76.0	13.6	11.5	12.1	37.2	33.8	35.2	1.4
30	11.6	72.9	13.5	13.3	12.0	38.8	35.4	38.3	2.9
31	12.1	76.9	13.4	13.2	10.7	37.3	33.9	35.2	1.3
32	8.0	75.0	13.2	8.2	12.3	33.7	30.3	35.5	5.2
33	10.7	74.8	13.0	13.2	12.3	38.5	35.1	36.7	1.6
34	13.9	72.6	12.8	12.7	11.9	37.4	34.0	26.8	−7.2
35	11.3	75.3	12.6	13.4	12.3	38.3	34.9	38.0	3.1
36	11.6	74.1	12.4	13.3	12.2	37.9	34.5	31.7	−2.8
37	10.4	71.0	12.2	12.9	11.7	36.8	33.4	32.6	−0.8
Totals...	....	....	447.6	445.1	448.7	1341.4			

* Less 20.0 for each functional reading.

Table 64 is worked out just as was Table 57. Thus the data for the first observation show values of 0, 9.6, and 74.8 for X_2, X_3, and X_4, respectively. Looking up the corresponding values in Tables 61, 62, and 63, gives values of 27.4, 31.7, and 32.3, for the three functional values. Subtracting 20 from each value, to reduce the subsequent clerical work, 7.4, 11.7, and 12.3 are entered in the functional columns. The three functional values are then added, and the sum entered in the 7th column. The entries for the functional readings are completed as shown, and the sum computed for each observation. Then the average of the 7th column is determined, giving the value 35.30. As the average of X_1 is 31.916, the value of the new constant, $a''_{1.234}$, is found by equation (58) to be

$$a''_{1.234} = 31.916 - 35.300$$

$$= -3.384$$

Accordingly, 3.4 is subtracted from each of the $\Sigma(f's)$ values to give the estimated value of X_1, X'''_1, which is then entered in the 8th column of Table 64.

The final step in computing the table is to subtract each of the estimated values, X'''_1, from the actual value X_1, giving the residuals z''', which appear in the last column.

Comparing the new residuals, z''', with the previous ones, z'', given in Table 58, we find that the size of the residuals has been increased in just about as many cases as it has been decreased. But when we compute the standard deviation of the new residuals, we find that the standard deviation of z''' is 2.80 bushels, or slightly smaller than the standard deviation of z'', 3.0 bushels.

Correcting the curves by further successive approximations.—The process ordinarily would be carried through one or more additional approximations by repeating the steps shown. Thus the last residuals, z''', when averaged and plotted with respect to the second set of approximation curves, would indicate whether any further modifications were needed in the curves; if any were made, new readings would be made from the new curves, new estimates of X_1 obtained from them, and another set of residuals determined. So long as the standard deviation of each new set of residuals is smaller than that of the previous set, the approximation curves may be regarded as approaching closer and closer to the underlying true curves. When, however, the curves have been determined as closely as is possible from the given data, the standard deviation of the residuals will show no further decrease, and may even increase slightly. In such case the set of curves showing the lowest standard deviation of residuals may be regarded as the final curves determined by the process.[13]

Stating the Final Conclusions.—After the final shape of the several net regression curves has been determined, it still remains to state those curves in such shape that their meaning is perfectly clear. The several functions may be stated to show the value of the dependent factor associated with given values of the particular independent factor when values of other independent factors are held at their mean. There are two alternative ways of stating the associated values: (1) as actual values; and (2), as deviations from the values otherwise expected.

To state the associated values as actual values, the following procedure may be used:

First, the mean of all the values read from the final curve is determined. For $f(X_2)$, this mean may be designated $M_{f(2)}$. The values from the curve are read off for selected intervals of X_2. Then the estimated values of X_1 for each of these values of X_2 (with values of X_3, X_4, etc., at their means) are determined by subtracting the mean of the curve readings from each of these actual readings, and adding to the result the mean of X_1. That is, if we use $X_1 = F(X_2)$ to designate these values of X_1, estimated from the net curvilinear relation to X_2, we can define them by the equation

$$X'_1 = F(X_2) = f(X_2) - M_{f(2)} + M_1 \qquad (60)$$

If, however, the expected values of X_1 for given values of X_2 are to be stated merely as deviations from the values which would otherwise prevail, those deviations may be determined by subtracting from each curve reading the mean of all the curve readings. If we use $F(x_2)$ to designate these expected deviations from the values which would otherwise prevail, we may define them by the equation

$$x'_1 = F(x_2) = f(X_2) - M_{f(2)} \qquad (61)$$

It is evident, from equations (60) and (61), that

$$F(X_2) = F(x_2) + M_1$$

In the actual statement of the results of a correlation study, it is frequently desirable to state the relation of the dependent factor to the most important independent factor according to equation (60), and to state the relation for the remaining independent factors according to equation (61). When that is done, the estimated values of X_1, based on all the independent factors, may be readily computed by taking the estimate from the most important factor, and then adding to or subtracting from that the corrections to take account of the departures of other factors from their means. Using X'_1 to designate this final estimate of the value of X_1, and taking X_3 as the most important factor, the estimate is given by the equation

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[13] In very exact work, the effect upon the residuals of modifications in each curve separately might be tested after this point, to insure that each individual regression curve had been fitted to the data with the greatest degree of accuracy.

$$X'_1 = F_2(x_2) + F_3(X_3) + F_4(x_4) + \ldots + F_n(x_n) \qquad (62)$$

The process of working out these final statements of the net curvilinear regression lines may be illustrated by the data of the corn yield problem. Since the rainfall (X_3) was apparently the most important factor, that may be taken as the one for which the regression is to be stated according to equation (60). If we regard the second approximation curve shown in Figure 38 and Table 62 as the final curve, then Table 64 gives the readings from this curve for each of the individual observations.

The mean of the readings of $f(X_3)$ is next computed from the values of Table 64. The sum of the 38 $f''(X_3)$ readings is 445.1, so

$$M_{f(X_3)} = \frac{445.1}{38} = 11.71$$

The mean value of X_1 is $M_1 = 31.92$. From equation (60),

$$F(X_3) = f(X_3) - M_{f(3)} + M_1$$

which is

$$F(X_3) = f(X_3) - 11.71 + 31.92$$
$$= f(X_3) + 20.21$$

✓All that is necessary, therefore, is to add the new constant, 20.2 to the values read from the curve. This process is shown in Table 65.

The computation for $F(x_4)$ follows the same form as that for $F(X_3)$, save that equation (61) is used instead, and hence the mean of X_1 is not involved. First the mean of all the readings for $f(X_4)$, as shown in Table 64, is computed, giving the value 11.81. The values for $F(x_4)$ are therefore given by the equation

$$F(x_4) = f''(X_4) - M_{f(X_4)}$$
$$= f''(X_4) - 11.81$$

These values are worked out in the next table, 66.

TABLE 65

Computation of Average Yield of Corn With Varying Rainfall, Holding Trend in Yield and Influence of Temperature Constant

Inches of rainfall, X_3	Readings from final curve,* $f''(X_3)$	Constant, $M_1 - M_{f(3)}$	Average yield, $F(X_3)$
7	6.0	20.2	26.2
8	8.2	20.2	28.4
9	10.5	20.2	30.7
10	12.3	20.2	32.5
11	13.4	20.2	33.6
12	13.2	20.2	33.4
13	13.0	20.2	33.2
14	12.7	20.2	32.9
15	12.5	20.2	32.7
16	12.3	20.2	32.5

* Curve readings minus 20, just as entered in Table 64.

TABLE 66

Computation of Deviation of Corn Yields from Yields Otherwise Expected, Because of Differences in Temperature for Season

Average temperature, X_4	Readings from final curve,* $f''(X_4)$	Constant, $M_{f(4)}$	Correction to expected yield, $F(x_4)$
70.0	11.6	−11.8	−0.2
71.0	11.7	−11.8	−0.1
72.0	11.8	−11.8	0
73.0	12.0	−11.8	0.2
74.0	12.2	−11.8	0.4
75.0	12.3	−11.8	0.5
76.0	12.1	−11.8	0.3
77.0	10.5	−11.8	−1.3
78.0	8.3	−11.8	−3.5

* Curve readings minus 20, just as entered in Table 64.

The net correction in the estimated yield to allow for the influence of trend can be obtained by carrying through a similar computation for

$F(x_2)$. The readings for $f''(X_2)$ sum to 447.6, so $M_{f(2)} = 11.78$. The values of $F(x_2)$ are then given by the equation

$$F(x_2) = f''(X_2) - 11.78$$

This computation is carried out in Table 67.

TABLE 67

COMPUTATION OF DEVIATION OF CORN YIELDS FROM THOSE OTHERWISE EXPECTED, BECAUSE OF NET TREND IN YIELDS

Number of year, X_2	Date	Readings from final curve,* $f''(X_2)$	Constant, $M_{f(2)}$	Correction to expected yield, $F(x_2)$
0	1890	7.4	−11.8	−4.4
5	1895	9.5	−11.8	−2.3
10	1900	11.0	−11.8	−0.8
15	1905	12.0	−11.8	0.2
20	1910	12.7	−11.8	0.9
25	1915	13.5	−11.8	1.7
30	1920	13.5	−11.8	1.7
35	1925	12.6	−11.8	0.8

* Curve readings minus 20.

The conclusions of the study can then be stated as shown in the last column of each of the last three tables, free from all of the previous details.

The relations for each of the variables can also be combined to show the expected or estimated yield for various combinations of the independent factors. Thus for the present case, it might be desired to combine the findings into a table showing the expected or probable yield for any given combination of rainfall and temperature, with the 1927 trend of yield. These values can be obtained by taking the trend correction for 1927, +0.4, and combining it with the estimated influence of various quantities of rain and degrees of temperature. These estimates would then be defined by the equation

$$\begin{aligned} X'_1 &= F(x_2) + F(X_3) + F(x_4) \\ &= 0.4 + F(X_3) + F(x_4) \end{aligned}$$

Combining the readings for $F(X_3)$ from Table 65 with those for $F(x_4)$ from Table 66, and adding in the correction for $F(x_2)$ as just stated, estimated yields are obtained as shown in Table 68.

TABLE 68

Estimated Yield of Corn, in Bushels per Acre, With Varying Rainfall and Temperature Conditions, for 1927

Inches of rainfall *	Average temperature †				
	70°	72°	74°	76°	78°
7	26.4	26.6	27.0	26.9	23.1
9	30.9	31.1	31.5	31.4	27.6
11	33.8	34.0	34.4	34.3	30.5
13	33.4	33.6	34.0	33.9	30.1
15	32.9	33.1	33.5	33.4	29.6

* Total for June, July, and August; average for 9 Corn-Belt stations.
† Average for June, July, and August, at same 9 stations.

By combining a table such as Table 68 with a statement of the extent to which yields averaged higher or lower than those shown at different times through the period, all of the conclusions from the study can be presented in simple form, easy to understand.

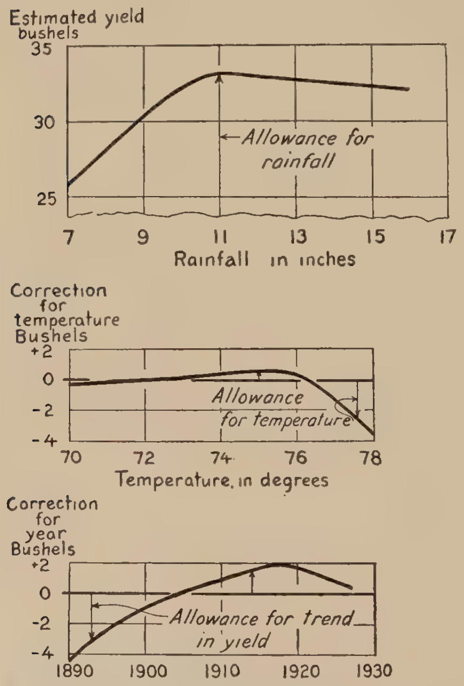

Fig. 40.—Relation of yield of corn to rainfall, temperature, and time.

The final results of curvilinear correlation studies, after being simplified to the form shown in Tables 65 to 67, or in Table 68, may also be expressed graphically for final publication. Thus all three relations might be combined into a single figure, such as shown in Figure 40, to present in relatively simple form the final conclusions reached by the statistical analysis.

It might be noted at this point that Table 68 is much more than merely a table of average yields for various rainfall and temperature groups. There were only 38 observations to begin with, and only 14 of those were under 74 degrees temperature. If these 14 observations had been grouped according to year and rainfall, and the average yield determined for each class, only the roughest sort of groups could have been made, and even then the averages would have had but little reliability. As the result of the correlation study, however, all 38 observations have been drawn on to determine the relations. The table shows the yield most likely to be received with any of 25 different combinations of rainfall and temperature, for the trend in 1927. Other estimates could be shown for an infinite number of other combinations. Furthermore, it is known that estimates made from such tables agreed with the actual yields to within 2.8 bushels in about two-thirds of the original cases. The reliability of these estimated yields is thus greater than it would be for any average of a few cases alone. This example illustrates the ability of correlation analysis both to bring out of a series of observations relations which are not observable on the surface and to provide a basis for estimating the probable effect on the dependent factor of new combinations of the independent factors.

These results also serve as a basis for estimating yields in other years. Thus in 1928 the rainfall was 14.9 inches, and the temperature 72.3 degrees, at the same stations. Interpolating from Table 65, the indicated yield would be 32.7 for rainfall, plus 0.1 correction for temperature from Table 66. Extrapolating the trend line gives 0.3 as the trend correction, making a total estimated yield of 33.1 bushels for 1928. The actual average yield for the 6 states was 34.9 bushels. The estimated yield is thus only 1.8 bushels from the actual yield, or about as close as would be expected with a standard deviation of 2.8 for the estimates for the 38 years included in the study. In 1929, with a rainfall of 10.6 inches and a temperature of 73.3 degrees, the correlation results give an estimated yield of 33.6 bushels, as contrasted with an actual yield of 31.2.

In this particular case the final shapes of the regression curves showing the net differences in yield with differences in rainfall and time are not greatly different from those indicated by simple correlation. In some cases, however, the final shape of the curves may be markedly different from the apparent shape before the variation associated with other factors has been eliminated. Thus the final shape of the curve showing the net differences in yield with differences in temperature, after allowing for the influence of rainfall and time, is quite different from what might have been expected from the original observations, as is illustrated in Figure 41. The curvilinear net regression is also quite different from the linear net regression, indicating that 74 to 76 degrees is the optimum temperature, whereas the straight line indicated that the lower the temperature, the higher the yield. With multiple correlation, as with simple correlation, the determination of the regression curves makes the results much more definite, adequate, and usable, than does merely the determination of the linear regressions.

Limitations on the Use of the Results.—It should be noted that the results of the corn yield analysis apply only to the same area from which the data were drawn, and to the period which they covered. Thus they provide no basis for estimating corn yields in other sections, and their use in estimating yields in other periods—as in subsequent years—is attended by increasing risk due to the necessity of extrapolating the trend regression. While this may give fair results for a year or two, as has been illustrated, it may tend to become increasingly inexact. For example, it may be that the trend of yield did not really turn downward about 1920, but only flattened out—additional years of observations will be needed to really tell which is correct.

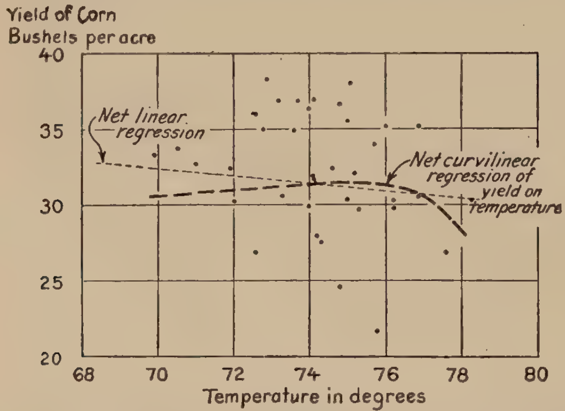

Fig. 41.—Comparison of apparent relation of corn yields to temperature with net relation after eliminating influence of rainfall and of trend in yield.

Other multiple curvilinear correlation studies illustrate other limitations to the application of the results secured. Thus in a study of the price of eggs in New York City, records were secured during a period of a few days on the retail sales prices of each of a number of dozens of eggs, and of the size, color, and quality of the eggs. (The data are given in the problem in Chapter 17.) By determining the net regression of price upon each of the factors, using the method illustrated, the net change in egg prices with changes in each of these factors can be determined. But it is readily apparent that size, quality, and color are not the only factors which might cause egg prices to vary. Prices change from one time of year to another, because of changes in seasonal demand, in supplies on the market, and in response to other factors as well. Prices also vary from place to place on the same day, and even at different stages in the marketing process in the same city on the same day—between sales at wholesale and retail, for instance. When we say that the results of the egg price study enabled us to estimate egg prices to within 5 cents two-thirds of the time, it must be remembered that the statement only holds true for *the same universe from which the original samples were selected.* In this case the samples were all selected from sales at retail, in the New York metropolitan area, in a particular period. The results therefore apply only to the reasons for variations in egg prices between particular stores, in that particular city, in that particular period. They might indicate the effect of similar differences in quality or weight on prices from store to store in the same city at other times of the year, or in other cities; but we could not be certain of that from this material alone. Other studies, covering those other ''universes,'' would be needed to prove or disprove that supposition; for the conclusions, of and by themselves, offer no statistical evidence except for their own particular ''universe.'' For that reason, each of the final tables should indicate clearly the conditions to which its conclusions apply, and thus definitely limit the statistical statement of the results to the particular conditions which they really represent.

Summary.—This chapter has discussed methods of determining curvilinear multiple regressions. These show the extent to which changes in the dependent variable are associated with changes in each particular independent variable, while simultaneously removing that part of the variation in the dependent variable which is associated (linearly or curvilinearly) with other independent variables. A method of determining the curves by successive graphic approximations is presented step by step. Since this method does not involve making definite assumptions as to the final shape of the curves, it is to be preferred to more mathematical methods, presented in a subsequent chapter, unless there is a logical basis for the choice of specific functions. Methods of simplifying the conclusions for popular statement are illustrated, and the universe to which they are applicable is briefly considered.

CHAPTER 15

MEASURING ACCURACY OF ESTIMATE AND DEGREE OF CORRELATION FOR CURVILINEAR MULTIPLE CORRELATION

In presenting linear multiple correlation it was pointed out that coefficients could be computed to show (1), how closely estimated values of the dependent variable, based on the linear regression equation, could be expected to agree with the actual values; and (2), what proportion of the total observed variation in the dependent factor could be explained or accounted for by its relation to the independent factors considered. These coefficients were, respectively, the standard error of estimate and the coefficient of multiple correlation. Exactly parallel coefficients can be computed to show the significance of curvilinear multiple correlation, employing curvilinear net regressions such as those discussed in Chapter 14. The term ''standard error of estimate'' is again used to indicate the measure of the probable accuracy of estimated values of the dependent factor; but, following the usage in simple curvilinear correlation, the term ''index'' is used to denote the fact that curvilinear regressions have been employed, and hence the proportion of variation accounted for is shown by the ''index of multiple correlation.''

Standard Error of Estimate.—In working through the various steps in determining the net regression curves by the method of successive approximations, in Chapter 14, the estimated values were subtracted from the actual values for each observation, and the resulting residual values, z'', z''', etc., were obtained. The standard deviations of these residuals were used as an indication of the accuracy of estimate for each set of curves. Where a very large number of observations is employed, such standard deviations of the residuals may be regarded as an indication of the extent to which estimated values of the dependent variable made from new sets of observed values drawn from the same universe may be expected to agree with the actual value of the dependent variable. Thus if we use $S_{1.f(2,3,4)}$ to designate the standard error of estimates of X_1, made on the basis of curvilinear relations to X_2, X_3, and X_4, and $z_{1.f(2,3,4)}$ to represent the residuals obtained using the final curvilinear regressions to estimate the dependent factor, the standard error may be defined by the equation

$$S_{1.f(2,\,3,\,4)} = \sigma_{z_{1.f(2,\,3,\,4)}} \qquad (63)$$

If the standard error of estimate for the final regression curves for the egg price problem mentioned in the previous chapter were 5 cents, that would mean if other purchases of eggs had been made in the same territory on the same day, it would have been possible to estimate the price to be paid for each dozen from their physical characteristics, to an accuracy indicated by that standard error. Two-thirds of the estimated values would probably have fallen within a range of 5 cents of the prices actually charged.

With the corn yield problem, the standard deviation of the residuals from the last set of curves was 2.8 bushels. In this case no other "sample" can be drawn from the same "universe" except those included in the problem, for the universe was restricted to the years studied, 1890 to 1927. Extrapolating the trend line, however, it is fairly safe to say that estimates made for the same region for subsequent years can be expected to have a standard deviation of at least 2.8 bushels. If the trend used did not prove correct for subsequent years, the errors might be considerably larger.

The relation shown in equation (63) holds exactly true only where there are a very large number of cases included in the sample dealt with. Where the sample is no larger than is usually available to the research worker, there is a tendency for the standard deviation of z to be somewhat smaller than the standard error which will prevail when estimates are made for new cases drawn from the same universe. The smaller the number of observations, the larger the number of independent variables included, and the more complex the curves employed, the greater will be the tendency for the observed standard deviation to underestimate the true standard error. This may be illustrated by results from an experimental study of the stability of multiple curvilinear correlation results. In this case a universe of known correlation was employed, and successive samples were drawn of various sizes, repeating each drawing a number of times for each size sample. The curvilinear regressions were then determined for each sample separately by the successive approximation method, and the standard deviations worked out for the residuals in each case. The entire analysis was then repeated, employing a universe of a higher correlation. The central values of these standard deviations of the residuals, for each size of sample, were as follows:

Number of observations	Observed standard deviation of z *	
	Universe 1	Universe 2
30	1.95	1.53
50	2.18	1.64
100	2.21	1.72
Entire universe	2.40	1.80

* These values are the median values observed.

It is quite evident from these results that the samples tended to give standard deviations smaller than that which actually was true for the universe as a whole, and further, that the smaller the sample employed, the greater the overestimate of the reliability of the estimated values.

It is therefore necessary to correct the observed σ_z to give $\bar{S}_{1.f(2,\,3,\,\text{etc.})}$. This correction is given in the following equation:

$$\bar{S}^2_{1.f(2,\,3,\,\text{etc.})} = \frac{\sigma^2_{z1.f(2,\,3,\,\text{etc.})}}{1 - m/n} \tag{64}$$

or

$$\bar{S}^2_{1.f(2,\,3,\,4,\,\text{etc.})} = \frac{n\sigma_z^2}{n - m} = \frac{\Sigma(z^2)}{n - m} \tag{65}$$

Where n = number of observations in the sample,
and m = number of constants represented (either mathematically or graphically) in the regression equation.

It will be seen that equation (65) is exactly similar to equation (42) for the standard error of estimate in linear multiple correlation problems. For curvilinear problems, however, the value m has a somewhat different meaning. Thus in the experimental results just discussed, three independent factors were involved, so the regression equation was of the form

$$X_1 = a + f(X_2) + f(X_3) + f(X_4)$$

The corresponding linear regression equation would involve only four constants, so m would be equal to 4. For curvilinear regressions, however, at least two constants would be necessary to represent each regression curve, and possibly more. In the experimental study each curve had only one bend, either upward or downward. It was judged, however, that the curves could not be represented by second-order parabolas, since their shapes did not follow the smooth symmetrical curve which that type of function is capable of describing. Instead, it was judged that a third order parabola would be necessary to give a fairly satisfactory fit to each regression curve. The conclusion was therefore reached that three constants would be necessary for a mathematical representation of each regression curve. On that basis the entire regression equation would represent approximately ten constants, 3 for each of the three curves, and one for the value *a*.

/M

Using 10 for m in equation (64), the value of $\bar{S}_{1.f(234)}$ for the smallest sample shown in the statement on page 265 may be worked out as follows:

$$\bar{S}^2_{1.f(234)} = \frac{\sigma_z^2}{1 - \frac{m}{n}} = \frac{(1.95)^2}{1 - \frac{10}{30}} = \frac{3.80}{0.667}$$

$$\bar{S}^2_{1.f(234)} = 5.70$$

$$\bar{S}_{1.f(234)} = 2.39$$

It is evident that this corrected value is much closer to the true value for the entire universe, 2.40, than was the original standard deviation of z.

Carrying the same correction through for the other values shown on page 222, standard errors of estimate are obtained as shown in the following statement.

Number of observations n	Value used for m	Universe 1		Universe 2	
		Observed σ_z	Calculated $\bar{S}_{1.f(234)}$	Observed σ_z	Calculated $\bar{S}_{1.f(234)}$
30	10	1.95	2.39	1.53	1.87
50	10	2.18	2.43	1.64	1.83
100	10	2.21	2.33	1.72	1.82
Entire universe	....	2.40	....	1.80	....

The superior accuracy of the corrected values is evident throughout this table—in each case they come much nearer to agreeing with the true value for the universe than do the uncorrected values.

Using equation (64) to obtain the standard error of estimate for the corn yield problem, it is first necessary to decide on the value to use for m. That problem also employed three independent variables, just as did the experimental study, and the final σ_z was 2.80 bushels. Although none of the three regression curves has more than one bend, none of them is of the symmetrical shape that can be described by the parabola; instead, at least a cubic parabola would be described to represent the curves for $f(X_3)$ and $f(X_4)$, while probably a quantic parabola, involving four constants, would be required to represent $f(X_2)$ with its final shape, or three constants with its first form. The final regression equation for corn yields might therefore be assumed to represent one constant for a, four for $f(X_2)$, three for $f(X_3)$, and three for $f(X_4)$, or a total of eleven in all. Inserting this value and the number of cases in formula (64), it becomes

$$\bar{S}^2_{1.f(234)} = \frac{\sigma_z^2}{1 - \frac{m}{n}} = \frac{(2.80)^2}{1 - \frac{11}{38}} = 11.98$$

$$\bar{S}_{1.f(234)} = 3.46$$

Although the standard deviation of the observed residuals was only 2.8 bushels, this standard error of estimate indicates that in using the results in making estimates for other years, the accuracy is likely to be less, even though the trend line is correctly extended. Instead of the estimated values probably coming within 2.8 bushels of the actual values in 68 per cent of the cases, they are likely to come that close in only about 58 per cent of the estimates, and an error of 3.5 bushels would have to be allowed to take in 68 per cent of the cases. In this particular problem, with three regression curves determined from 38 observations, the correction embodied in equation (64) is important. If the same set of conclusions had been obtained from 20 observations, with the same standard deviation of the residuals, applying the correction formula would have increased the standard error of estimate to above 4.1 bushels, illustrating again the tendency of a small sample to exaggerate the accuracy of estimate.

Index of Multiple Correlation.—The coefficient of multiple correlation, it will be remembered, indicated the proportion of the total variation in the dependent factor which could be accounted for on the basis of the linear relations to the several independent factors. In exactly the same way the proportion of variation which can be accounted for on the basis of the curvilinear relations to the several independent factors is termed the "index of multiple correlation," and is designated by the term P, that is, capital *rho*. Following the definition, and using X''_1 to indicate values of X_1 estimated from the other factors on the basis of the net curvilinear regressions, the index of multiple correlation may be defined by the equation

$$P = \frac{\sigma_{X''_1}}{\sigma_{X_1}}$$

It may be most accurately computed, however, by making use of the standard deviations of the residuals. Using z'' to represent $X_1 - X''_1$, then

$$P^2 = 1 - \frac{\sigma^2_{z''}}{\sigma^2_{X_1}}$$

With small samples $\sigma_{z''}$ tends to be smaller than the actual standard error of estimate in the universe as a whole. For that reason, the index of correlation, as computed by the formula just given, tends to exceed the correlation that actually obtains in the universe from which the observations are drawn. Data from the experiment mentioned earlier illustrate this point. The following tabulation shows the modal index of multiple correlation for each size of sample, in comparison with the true index of correlation for the entire universe.

Number of observations in sample	Observed index of multiple correlation in samples drawn from same universe
30	0.77
50	0.71
100	0.68
Entire universe	0.62

In every case the observed correlation exceeds the true correlation in the universe, and the smaller the size of the sample, the larger the difference. It is therefore necessary to apply to the index of multiple correlation the same type of correction which was applied in obtaining the standard error of estimate. This may be done either by substituting the adjusted standard error of estimate for the observed standard deviation of the residuals in the equation to determine P, or by making the adjustment directly in the equation itself. The following formulas show both methods.

$$\left.\begin{aligned} \bar{P}^2_{1.234} &= 1 - \left[\left(\frac{\bar{S}^2_{1.f(234)}}{\sigma_1^2}\right)\left(\frac{n-1}{n}\right)\right] \\ \bar{P}^2_{1.234} &= 1 - \left[\left(\frac{\sigma^2_{z_{1.f(2,3,4)}}}{\sigma_1^2}\right)\left(\frac{n-1}{n-m}\right)\right] \\ &= 1 - \left[\left(\frac{\Sigma(z^2)_{1.f(234)}}{\Sigma x_1^2}\right)\left(\frac{n-1}{n-m}\right)\right] \end{aligned}\right\} \quad (66)$$

The adjusted indexes of multiple correlation work out for the experimental data as shown in the following statement:

Number of observations, n	Value used for m	Crude, P	Adjusted, $\overline{P}$
30	10	0.77	0.64
50	10	0.71	0.63
100	10	0.68	0.65
Entire universe	....	0.62	

Here again the adjusted values are found to be in much better agreement with the true value for the entire universe than are the crude values. For that reason equation (66) should always be employed in calculating the index of multiple correlation.

Unless the index of multiple correlation, as calculated with the adjustment, is larger than the coefficient of multiple correlation, with its comparable adjustment by equation (47), there is no statistical evidence of significant curvilinearity in the regression lines. Unless the standard error for the curves is lower even after adjustment, any reduction in the unadjusted standard deviation of $z_{1.f(2,\,3,\,4)}$ as compared with σ_z from the linear regression, would be merely a fictitious improvement in accuracy. Such an apparent improvement might result from the greater number of constants represented in the regression curves, giving the same effect as if additional variables had been added to those taken into account.

Once the index of multiple correlation has been computed by equation (66), the square of its value may be employed to represent the total determination, i.e., to measure the proportion of the total variance in X_1 which can be accounted for on the basis of the curvilinear relations to the several independent factors. To maintain the same terminology, this may be termed the *index* of total determination, to distinguish it from the *coefficient* of total determination, which applies to linear multiple correlation.

The computation of the index of multiple correlation may now be illustrated from the data of the corn yield problem. In that study the original standard deviation of the yields was 4.30 bushels, the standard error of estimate by linear multiple correlation, 3.87 bushels, and the coefficient of multiple correlation, after adjusting for the number of cases, 0.49. The standard error of estimate for the final regression curves, as worked out on page 224 was 3.46 bushels. Computing the index of multiple correlation by equation (66), in its first form, we have

$$\bar{\mathrm{P}}^2_{1.234} = 1 - \frac{\bar{S}^2_{1.f(234)}}{\sigma_1^2}\left(\frac{n-1}{n}\right)$$

$$= 1 - \frac{(3.46)^2}{(4.30)^2}\left(\frac{37}{38}\right)$$

$$= 0.369$$

$$\bar{\mathrm{P}}_{1.234} = 0.61$$

The index of multiple correlation is therefore 0.61, as compared with the coefficient of multiple correlation of 0.49. The total determination, which was 24 per cent for the linear relation, has been raised to 37 per cent for the curvilinear. The increase indicates that the linear relations did not express all of the effect of the three independent variables, and that taking the curvilinearity of the regressions into account has added significantly to the importance of the factors considered. With such a low determination, however, it is evident that there are other perhaps more important factors not yet taken into account.

Measuring the Net Curvilinear Importance of Individual Factors.—No method has been devised as yet to determine the portion of the index of total determination which can be ascribed to each of the several independent factors, solely from the methods used in obtaining the several regression curves themselves. The final slope and shape of the curves may be tested, however, by correlating the curve readings for each observation with the original values of the dependent factor, so as to obtain the partial regression coefficients indicated in equation (80), and explained in Chapter 21.

$$X_1 = a' + b_{12'.3'4'}[f(X_2)] + b_{13'.2'4'}[f(X_3)] + b_{14'.2'3'}[f(X_4)]$$

If that is done, the coefficient of multiple correlation, $R_{1.2'3'4'}$, measures the total correlation with respect to the several curvilinear functions (including the final adjustments), and is therefore the index of multiple correlation, $\mathrm{P}_{1.234}$. It is, however, still subject to the same adjustment for number of constants as are indexes of multiple correlation computed in other ways, and should therefore be corrected as follows:

$$\bar{\mathrm{P}}^2_{1.234} = 1 - (1 - R^2_{1.2'3'4'})\left(\frac{n-1}{n-m}\right) \tag{67}$$

Indexes of partial correlation, or of part correlation, can be determined with respect to the curvilinear regressions of the several independent variables, as shown in equation (80), in exactly the same way that the parallel coefficients of correlation are obtained. Since the curvilinear transformation relates solely to the net regression of X_1 on each of the independent variables, the meaning of the partial indexes with respect to the separate variables is open to some doubt. The index of part correlation would retain its parallel meaning, however, measuring the correlation of each independent factor, according to its net regression curve, with the dependent factor after correcting it for variations in the several other factors from their means. This index would thus measure the closeness of correlation for such relations as are shown in Figures 37 to 39 of Chapter 14, and would have a concrete meaning. This correlation could be computed directly from the readings for the individual observations shown in Table 64, by calculating the two values

$$\Delta = X_1 - f(X_3) - f(X_4), \quad \text{or} \quad = z''' + f(X_2)$$
$$D = f(X_2)$$

then the linear correlation between Δ and D, $r_{\Delta D}$, would give the index of part correlation between X_1 and X_2, with X_3 and X_4 constant, or ${}_{12}\rho_{34}$. If only the indexes of part correlation were desired, application of this method to each of the variables in turn might be a faster method than the one previously described.

Since the same number of constants has been involved in the determination of each of these indexes of part correlation as in the index of multiple correlation, it is probable that they would need to be adjusted in the same manner, using equation (67). However, no experimental evidence is yet available on this point.

Summary.—For curvilinear multiple regression equations it is possible to obtain standard errors of estimate, indexes of multiple correlation, and indexes of separate or part determination, which serve the same purpose that the comparable coefficients serve for linear multiple regressions. Owing to the extent to which the process of fitting the curves may exaggerate the significance of the results, it is even more important to adjust the several measures with respect to the number of observations and numbers of constants involved than it is with linear multiple correlation.

CHAPTER 16

SHORT-CUT METHODS OF DETERMINING NET REGRESSION LINES AND CURVES

In problems where the correlation is fairly high, the number of variables is not too large, and the number of observations is relatively small (say not over 50 to 100 cases), net regression lines and curves may be determined by a combination of inspection and graphic approximation which takes only a fraction of the time required by the methods previously presented in detail. These methods have been only recently developed,[1] but they are very effective in practice.

The general basis of this method is to select, by inspection, several individual observations for which the values of one or more independent variables are constant, and then note the change in the dependent variable for given changes in the remaining independent variable. This process is repeated for additional groups of observations for which the other independent variable or variables are constant (or practically so) but at a different level than for the first group. The relation between the dependent variable and the remaining independent variable, as indicated by a series of such groups, approaches the *net* regression line or curve, since the cases have been so selected as largely to cancel out the variation associated with other independent variables. A first approximation line or curve is then drawn in by eye, and the residuals from this curve, measured graphically, are used to determine the regression for the next variable, again selecting cases so as to eliminate the influence of other independent variables. The final fit of the several lines or curves is tested by the same successive approximation process employed in Chapters 10 and 14. Since the initial lines or curves approach much more closely to the final net regressions, the process is much shorter, and fewer steps are required.

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[1] L. H. Bean, "Applications of a simplified method of graphic curvilinear correlation." Mimeographed preliminary report, U. S. Bur. Agr. Econ., April, 1929; and "A simplified method of graphic curvilinear correlation," Jour. Amer. Stat. Assoc., Vol. XXIV, pp. 386–397, Dec., 1929.

Linear Net Regressions.—The short-cut method for linear regressions may be illustrated by the same farm income problem utilized in Chapters 10, 11, and 12. The first step is to number each one of the observations as listed in the first four columns of Table 47, page 166, so that they may be distinguished from one another. The next step is to chart the values of the four variables for each observation in succession and connect them by lines just as if they were entries in a time series, as shown in Figure 42. (Classifying the records in order with respect to one of the independent factors before taking this step might be advisable.)

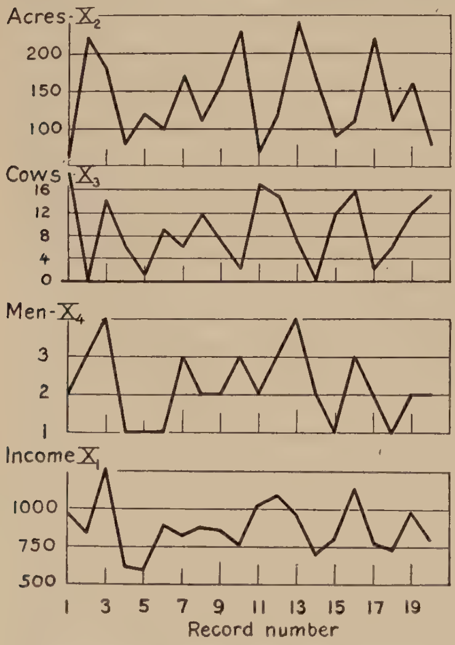

FIG. 42.—Acres, cows, men, and income, on 20 farms.

Comparing the different lines in Figure 42, it is seen that variation in incomes appears to be more closely associated with variations in cows than with either of the other factors. (Dot charts of X_1 with X_2, X_1 with X_3, and X_1 with X_4, might be used instead to reach this conclusion.) The relation of X_1 to X_3, number of cows, *for constant numbers of acres and men*, will therefore be examined first.

From Figure 42 we note that of the farms with the largest numbers of acres, farms 2 and 10 each have 3 men employed, while farms 13 and 17 have 4 and 2, respectively. Accordingly we plot the cows and incomes for these farms on a new dot-chart as shown in Figure 43, indicating the number of the farm represented by each dot. The placing of these dots does not seem to indicate any marked relation of income to the number of men; we therefore draw in a straight line free-hand, to fit approximately the change in income with changes in cows, as shown by these four observations. (The values may be taken from Table 69, page 235.)

Turning to the small farms, we note that farms 6, 15, and 18, each with between 90 and 110 acres, have 1 man apiece; and farms 8, 11, and 20, with 70 to 110 acres, have 2 men apiece. Plotting the corresponding observations as hollow dots on Figure 43, again there is little evidence of any influence of the differences in number of men. The other small farms, 4, 5, and 16, are accordingly plotted, and a line, estimated graphically to pass through the nine observations as well as possible, is drawn in as shown.

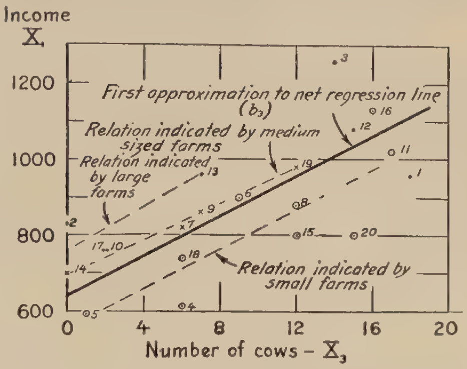

FIG. 43.—Income plotted against cows, on specified farms, and first approximation to net linear regression on cows.

Finally, it is noted that farms 7, 9, 14, and 19 all have between 150 and 170 acres, so these are plotted on Figure 43 as crosses, to distinguish them (the differences in the number of men are ignored at this step, since they have been found to have little apparent relation to the income in the previous cases) and a line is drawn through these last cases, as indicated.

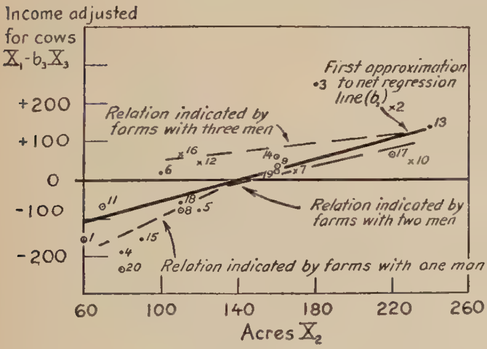

FIG. 44.—Income adjusted for cows (by first approximate regression), plotted against acres on specified farms, and first approximation to net linear regression on acres.

Comparing the three lines, it is seen that all have about the same slope, so a single line is drawn in to pass through the intersection of the averages of cows and of income, with a slope averaging the slope of the other three lines. This last line is the first approximation to the net regression of income on cows, with acres and men constant. The dots for the remaining farms, numbers 1, 3, and 12, are then plotted in, with the numbers to indicate their identity.

For the next step, a blank chart is prepared, as shown in Figure

44, to show the relation between acres, X_2, and the departures of income, X_1, from that expected on the basis of the approximate regression on number of cows. This chart is completed by scaling off the vertical departure of each observation in Figure 43 from the approximation line, and then plotting that departure in Figure 44 as a departure from the zero line, with the number of acres for the same observation as abscissa. The identity of the observation represented by each dot is again shown by its number. The relation of acres to income is now clearly evident (in fact, were this not a discussion of linear correlation, fitting a curve would seem to be justified). It is next noted that farms 4, 5, 6, 15, and 18 have but one man apiece. Accordingly a line is dotted in to pass as near the dots for these farms as possible. Farms 2, 7, 10, 12, and 16 have 3 men each, so a line is fitted to them graphically, as indicated. Farms 1, 8, 9, 11, 14, 17, 19, and 20 have 2 men each, so they are designated by enclosing each of them with a circle, and a line is fitted free-hand to them. All of these lines are of somewhat the same slope, so a final line is drawn in by eye, averaging the slope of the other lines, and intersecting the zero line at the abscissa corresponding to the average number of acres. This line is the first approximation to the regression of income on acres determined while holding constant the approximate effects of both cows and number of men.

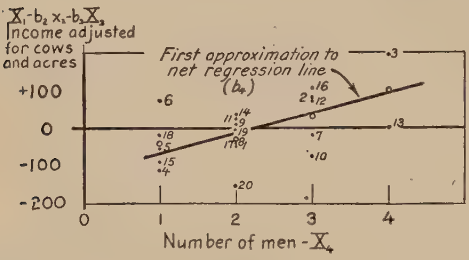

FIG. 45.—Income adjusted for cows and acres (by first approximate regressions), plotted against number of men on specified farms, and first approximation to net linear regression on men.

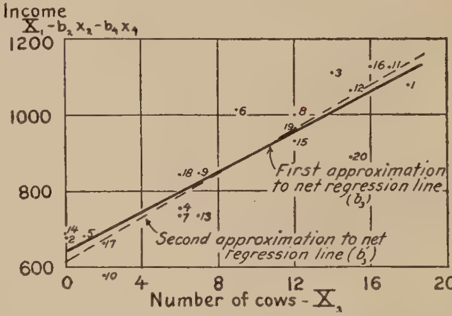

FIG. 46.—Income adjusted for acres and men (by first approximate regressions), plotted against cows, and first and second approximations to net regression on cows.

The next step is to prepare a chart for number of men and adjusted income, as shown in Figure 45. The deviations of the individual observations from the approximate regression line in Figure 44 are measured graphically, and plotted in as deviations from the zero line in Figure 45, with the number of men for each observation as abscissa. The placing of these dots indicates a tendency for income to increase with number of men. The average adjusted income for each number of men is determined by inspection, and indicated by the small circles. Then a straight line is fitted by eye so as to intersect the zero line at the average of X_4, and fit these averages as well as possible.

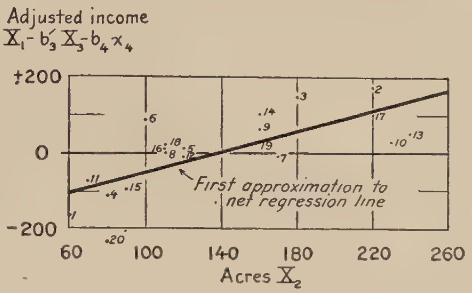

FIG. 47.—Income adjusted for cows (by second approximate regression) and men (by first approximation), plotted against acres.

The next step is to check the slope of the previous approximate net regression lines, to see if any changes are needed, now that the effect of other factors has been more accurately allowed for. To do this, the line from Figure 43 is drawn in on Figure 46. The deviations of each of the observations in Figure 45 are then scaled off graphically, and plotted in Figure 46 as vertical deviations from the line, with the number of cows, X_3, as abscissa. The plotting of these deviations indicates that a slightly steeper line might fit better, since it is found that although in the range 0 to 2 cows 2 dots fall below the line whereas 3 fall above, in the range 14 to 18, 4 out of 6 dots fall above the line, and in the range 6 to 8 cows, 3 of the 5 observations fall below the line. Accordingly a revised line is drawn in free-hand, passing through the intersection of the averages of cows and income as before, and fitting the new dots as well as possible. The first line for the regression of income on acres is then checked in the same manner, by plotting the deviations from the new line in Figure 46 as deviations from the first approximate regression on acres (Figure 44). This

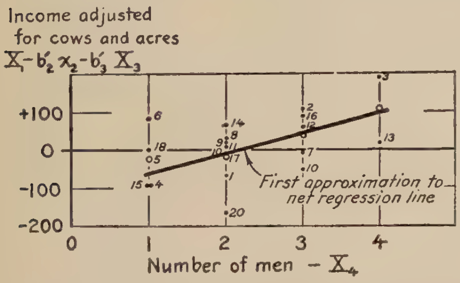

FIG. 48.—Income adjusted for cows and acres (by second approximate regressions), plotted against men.

process, carried out by graphic plotting just as before, is shown in Figure 47.

The distribution of the dots in Figure 47 shows that the observations are so nearly evenly balanced about the line now that no further change in the line is necessary. It is evident that a curve would fit better than the straight line, but for the present we are considering linear relations only.

Since no change has been made in the regression for X_2, all that remains is to check the first line for the regression on X_4, using the deviations from the line in either Figure 46 or in Figure 47. Plotting these deviations graphically as before, above or below a line with the same slope as in Figure 45, gives the result shown in Figure 48. Since this figure shows no significant change from Figure 45, the line is left unchanged, and the lines on Figures 46, 47, and 48 are accepted as giving the approximate values for $b_{13\cdot24}$, $b_{12\cdot34}$, and $b_{14\cdot23}$, respectively. Calculating from these lines the increases in income per unit change, they come out 29.2 dollars per cow, 1.34 dollars per acre, and 52.7 dollars per man, as contrasted to the exact values of 26.3, 1.21, and 50.3, worked out in Chapter 12. Although the values are not identical, they are quite close—so close, probably, that the differences between them have no statistical significance in view of the small number of observations on which they are based.

The estimated income may now be worked out for each farm, either by taking readings directly from each curve, or by substituting the approximate values found for the regression coefficients in equation (39) to determine a, and then working out the estimates mathematically. In either case, the correlation and standard error could be computed only by working out the estimated values, calculating the residuals and their standard deviation, and substituting those in equations (42) and (48). Using values read directly from the figures, the process of computing the estimates is shown in Table 69.

The standard deviation of the z's computed in Table 69 is 68.06. Substituting this value in equations (42) and (48), the standard error of estimate and the multiple correlation work out as follows:

$$\bar{S}^2_{1.234} = \frac{n\sigma^2_z}{n - m} = \frac{20(4632)}{16} = 5790$$

$$\bar{S}_{1.234} = 76.09$$

$$\bar{R}^2_{1.234} = 1 - \frac{\bar{S}^2_{1.234}}{\sigma_1^2}\left(\frac{n-1}{n}\right) = 1 - \frac{5790}{27276}\left(\frac{19}{20}\right) = 0.798$$

$$\bar{R}_{1.234} = 0.893.$$

TABLE 69

Calculation of Estimated Income from Linear Regressions Determined by Approximation Method

Number	X_2 Acres	X_3 Cows	X_4 Men	X_1 Income	$f(X)_2$	$f(X_3)$	$f(X_4)$	X'_1	z
1	60	18	2	960	−106	1134	−11	1017	− 57
2	220	0	3	830	+110	612	+42	764	66
3	180	14	4	1260	+ 56	1022	+94	1172	88
4	80	6	1	610	− 80	789	−62	647	− 37
5	120	1	1	590	− 26	641	−62	553	37
6	100	9	1	900	− 52	876	−62	762	138
7	170	6	3	820	+ 43	789	+42	874	− 54
8	110	12	2	880	− 39	964	−11	914	− 34
9	160	7	2	860	+ 29	818	−11	836	24
10	230	2	3	760	+123	670	+42	835	− 75
11	70	17	2	1020	− 93	1110	−11	1006	14
12	120	15	3	1080	− 26	1051	+42	1057	23
13	240	7	4	960	+136	818	+94	1048	−88
14	160	0	2	700	+ 29	612	−11	630	70
15	90	12	1	800	− 66	964	−62	836	− 36
16	110	16	3	1130	− 39	1080	+42	1083	47
17	220	2	2	760	+110	670	−11	769	− 9
18	110	6	1	740	− 39	789	−62	688	52
19	160	12	2	980	+ 29	964	−11	982	− 2
20	80	15	2	800	− 80	1051	−11	960	−160

The new standard error of \$76.09 compares with that of \$74.65 obtained by the regular least squares method, and the multiple correlation of 0.893 by the approximation method compares with the value 0.898 obtained by the more exact method. As indicated by these slightly lower coefficients the approximation method is not quite so precise, yet for most practical purposes the results are nearly the same.[2]

The Short-cut Method Applied to Curvilinear Regressions.—The most striking advantage of the short-cut method is its use in determining net curvilinear regressions. Since the method of successive approximations previously presented depends on the convergence of successive approximate curves, the short-cut method secures results which are exactly as reliable, at a great saving of time in the preliminary stages of the solution.

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[2] In fact, the differences between the values obtained by exact solution and those obtained by the approximation method are no larger than might readily occur by chance if the mathematical analysis were repeated on a second sample of the same size, to judge from the standard errors of the three regression coefficients, when computed by the methods explained in Chapter 18.

The procedure will be illustrated by a small problem of three variables, but the same method may be applied to larger problems equally well.

The data to be considered are as follows:

DATA FOR SHORT-CUT METHOD OF DETERMINING REGRESSION CURVES

Year	Index of cotton consumption X_1	Crop-year price of cotton* X_2	Index of general industrial activity X_3
	Per cent	*Cents per pound*	*Per cent*
1919	96	22.0	84
1920	95	24.7	86
1921	88	14.5	66
1922	99	18.2	87
1923	106	25.3	101
1924	89	30.6	94
1925	105	24.6	105
1926	109	19.7	108
1927	122	15.3	106
1928	107	20.4	111

* Adjusted for price-level changes.

These data may be used to determine how changes in price affected consumption, and how changes in business activity affected consumption. We will disregard the data for 1928 for the present, and use that year for subsequent checking.

Examining these data, we note that years of low consumption tend to correspond to years of high prices, and *vice versa,* but there are several years when this fails to hold true. In 1921 and 1922, prices were low and consumption was low; but general industrial activity was also very low. Likewise in 1925, prices were fairly high, and cotton consumption was fairly high, but business activity was also high. Apparently both factors affect consumption; which to treat first is a matter of indifference.

We may proceed by noting that in 1919, 1920, and 1922 industrial activity was around 85; in 1925, 1926, and 1927 it was around 106; the other years were isolated cases. Accordingly we may compare prices with consumption for the two groups of years stated. The effects of industrial activity are fairly constant for each group. Figure 49 shows this comparison with the years of the first group designated by hollow circles and those of the second group designated by crosses. Connecting the dots for each of the two groups, we find they each lie along a continuous curve. Next, plotting in solid dots for the remaining years, we note that the dot for 1921, when industrial activity was very low, falls much below the level of the other two curves; but that the dot for 1923, when industrial activity was about 100 per cent, falls higher than would be expected. The dot for 1924, however, falls low, due to rather low business activity—not so low, however as it was in 1919, 1920, and 1922.

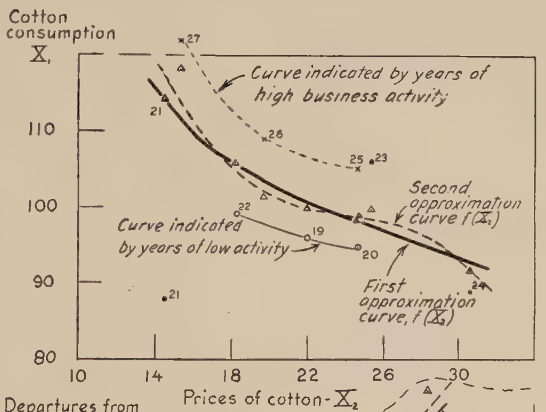

FIG. 49.—Cotton consumption and cotton prices in specified years, and successive approximations to net regression of consumption on price.

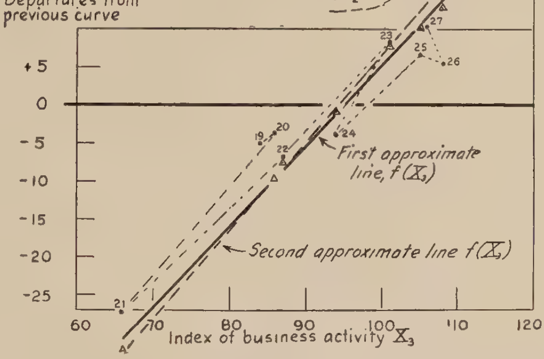

FIG. 50.—Cotton consumption adjusted for price plotted against index of business activity, and successive approximations to net regression of consumption on business activity.

A first approximation curve is then drawn in free-hand so as to pass approximately half-way between the two previous curves, but somewhat nearer the dot for 1924 than the lower curve as a whole. (This is temporarily ignoring the fact that 1923 appears out of line.) The departures of the actual quantities consumed from the quantities shown by this curve are then measured off graphically and plotted as ordinates with the accompanying values of X_3 as abscissas, as shown in Figure 50.

The position of the dots in Figure 50 indicates the strong positive relation between industrial activity and cotton consumption. On examining the dots by successive years, however, there seems some tendency to a secular change. This is particularly marked if successive years are connected by a continuous dotted line. There is a marked tendency for this line to shift to the right, with the last years standing out still farther to the right. Accordingly a straight regression line is drawn in, of the average slope of the lines connecting successive years. Since departures from the line are apparently associated with some independent factor changing with the passage of time, the deviations from the approximation line in Figure 50 may be plotted for successive years, to determine if there is a definite trend in consumption, after allowing for the relation to price and business activity. Figure 51 shows this comparison, with the departures from the line in Figure 50 plotted against time as abscissa. These residuals show a fairly continuous downward trend in time; therefore a straight-line trend is drawn through them, with its slope determined graphically, as the approximate net regression of consumption on time.

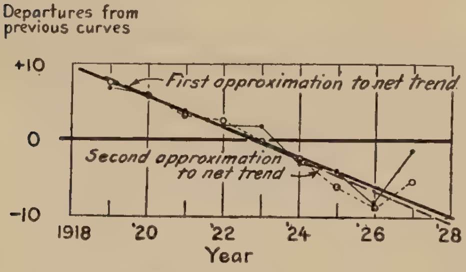

Fig. 51.—Cotton consumption adjusted for all other factors plotted against time, and successive approximations to net trend.

The accuracy of the previous regressions may now be tested by means of the residuals from this last regression, with all three factors allowed for at least approximately. Since there are so few cases this test can be made directly on the original charts, plotting the residuals from the line in Figure 51 as deviations from the curve of Figure 49, and using triangles to distinguish the new plottings. The location of these triangles suggests a slight modification of the shape of the first approximation curve; therefore a second approximation curve is drawn in. The departures (of the triangles) from this new curve are next used to check the accuracy of the first approximate regression on business activity. These residuals from the new curve for $f(X_2)$ are plotted as departures from the first approximation line for $f(X_3)$ in Figure 50, with values of X_3 as abscissas, and are indicated by triangles as before. A slightly steeper slope for the line is indicated. A new line is drawn in. The residuals (of the triangles) from this new line are then plotted as departures from the previous regression on time, to see if the changes in the other two lines will call for any further change here. Hollow dots are used to denote the new residuals. Apparently a slightly steeper net trend line is justified; this is accordingly drawn in.

The residuals from the final trend line might be again plotted against the other curves, to see if any further changes were necessary. Only for 1927, however, is the residual significant, and inspection of the other regression charts indicates that this observation falls in such a position that only a slightly better fit can be obtained. Only if a curve were employed for the regression of consumption on business activity would the residuals be reduced. In view of the small number of observations, and the very slight indications of curvilinearity, the employment of a curve does not seem justified in this case.

The closeness of the final regressions and the probable accuracy of estimate (assuming the economic conditions to remain the same as during this period) may now be determined by the usual methods. The estimated value of X_1 and the individual residuals are first computed by readings from the several regression curves, as follows:

Year	X_2	X_3	$f(X_2)$	$f(X_3)$	f(time)	X'_1	X_1	z
1919	22.0	84	100.0	−11.8	+8.0	96.2	96	−0.2
1920	24.7	86	98.7	− 9.8	+5.9	94.8	95	0.2
1921	14.5	66	117.5	−33.1	+3.8	88.2	88	−0.2
1922	18.2	87	104.8	− 8.5	+1.7	98.0	99	1.0
1923	25.3	101	98.5	+ 7.9	−0.4	106.0	106	0.0
1924	30.6	94	91.8	− 0.2	−2.5	89.1	89	−0.1
1925	24.6	105	98.8	+12.3	−4.6	106.5	105	−1.5
1926	19.7	108	102.0	+15.9	−6.7	111.2	109	−2.2
1927	15.3	106	113.3	+13.6	−8.8	118.1	122	3.9

The standard deviations of X_1 and of the residuals are next computed. The values are $\sigma_1 = 10.13$; $\sigma_z = 1.61$.

Since two linear regressions have been employed, and one curve which would require a cubic parabola to reproduce, the number of constants to state the regression equation mathematically would be 6.[3]

Only 9 observations have been employed, so this reduces $n - m$ to 3 cases. The correlation is so high, however, that even after this drastic correction the relations are shown to be significant, as is evident when equations (65) and (66) are worked through, as follows:

$$\bar{S}^2{}_{1.f(2,3,t)} = \frac{n\sigma_z{}^2}{n - m} = \frac{9(2.59)}{3} = 7.78$$

$$\bar{P}^2{}_{1.234} = 1 - \frac{\bar{S}^2{}_{1.f(2,3,t)}}{\sigma_1{}^2}\left(\frac{n-1}{n}\right) = 1 - \frac{7.78}{102.67}\left(\frac{8}{9}\right) = 0.933$$

[3] $X_1 = a + b_2X_2 + b'_2(X_2{}^2) + b''_2(X_2{}^3) + b_3X_3 + b_4t.$

Accordingly we may conclude that, under the same conditions as for the years included in the study, the consumption of cotton might be estimated from the three factors, cotton prices, industrial activity, and the net trend, with a standard error of 2.8 points; and that 93 per cent of the variation in cotton consumption has been associated with these three factors. These results may be tested by estimating the cotton consumption for 1928, a year not included in the analysis. For that year the price per pound was 20.4 cents, and the index of industrial production was 111. Substituting these values in the regression equation, and extending the net trend line to 1928, the estimate works out as follows:

$$\begin{aligned} X_1 &= f(X_2) + f(X_3) + f(t) \\ &= 101.2 + 18.1 - 10.9 = 108.4 \end{aligned}$$

The actual index of consumption for the year was 107, and the estimate comes out 108.4. The difference is well within the probabilities indicated by the standard error of 2.8. Apparently about the same economic conditions continued to operate in 1928 as in the years included in the study.

Application of the short-cut method to large samples.—The short-cut method might be applied to samples too large to plot the individual observations separately, by using a modification of the process of sub-grouping and averaging illustrated in Chapter 11. The averages from Table 42, plotted in Figures 30 and 31, indicated quite well the final slope of the net regression lines. That was because the influence of the other independent variable had been largely held constant by the process of sub-classifying. In the same way the lines of averages from sub-groups would tend to indicate the regression curves in problems where curves were needed. With a sufficient number of observations, the first approximation to each of the net regression curves might be obtained from charts of sub-averages similar to Figures 30 and 31. These several first approximation curves could then be made the basis for working out estimated values of X_1 and residuals. The process of successive approximations could then be continued exactly as illustrated in Chapter 14. Since the first approximation curves would approach fairly near to the true net regressions, the number of approximations required to obtain the same closeness of fit would usually be less than by the earlier method.

It should be noted that where two independent variables are very closely correlated, the composite effect of the two will tend to be thrown to whichever one is considered first by the graphic approximation method. But that is also in a smaller degree true with any of the other methods. As will be shown later (Chapter 18) the reliability of *any* net regression line or curve varies inversely with the extent to which the particular independent variable is correlated with the other independent variables. Where two variables are so closely correlated that the relation to the dependent variable may be ascribed to either or parceled out between the two, their individual effect is indeterminate. Only by securing a large enough sample can the true influence of each be judged. When a large enough sample cannot be secured, that is the inherent fault of the data, and not of the method employed. Any one of the several methods will tend to give results which are substantially the same—that is, which lie within the range of possibility imposed by the facts of the particular sample.

The time-saving graphic approximation method has been so recently developed that its possibilities and limitations have not yet been fully explored. The speed which it affords in multiple correlation analysis, however, bids fair to extend analyses of this type to many problems which were previously ruled out by the lengthy process required. It is undoubtedly true that care, understanding, and skill are necessary to obtain satisfactory results by the shorter method. Yet that is also true of the previous method—the indiscriminate and mechanical application of any analysis can lead only to mediocre and untrustworthy conclusions. In fact, the emphasis on critical study of the individual observations is a point in favor of the graphic short-cut method. It forces the researcher to know his data more thoroughly, and so to exercise thought and care in working out the relations and in interpreting their significance.

Summary.—Under certain conditions first approximations to multiple regression lines or curves may be obtained directly from the original observations by a graphic process based on the comparison of individual observations, considering several variables simultaneously. This process eliminates the necessity of computing linear regressions by arithmetical means. Further, it substitutes graphic measurements for arithmetic calculations in correcting these curves to their final shape by successive approximations. Carefully used, it materially reduces the time required in determining multiple regression curves.

CHAPTER 17

MEASURING THE WAY A DEPENDENT VARIABLE CHANGES WITH CHANGES IN A NON-QUANTITATIVE INDEPENDENT FACTOR

It is frequently desirable to determine the change in one variable associated with changes in an independent factor which varies in such a way that it cannot be measured quantitatively. Thus if the significance of various factors affecting farm values is to be determined, one may wish to include type of road as one of the factors, since a farm on a concrete road should be expected to be worth more than one on a dirt road, other factors being the same. Yet the designations, concrete, brick, macadam, gravel, and dirt, cannot be considered in the correlation analysis in the way that the numbers measuring variable factors are treated.

Where no other factors are involved, a non-quantitative factor may be treated by sorting with respect to that factor, and averaging the dependent variable. Thus if only type of road is being considered, the average value per acre of farms fronting on each type of road may be taken as the measure of the influence of roads on value. If, however, several other factors must be considered at the same time, such as value of improvements, productivity of the soil, distance from town, etc., and if there is any relation between differences in these factors and differences in road type (as in general there will tend to be) the influence of road type must be measured by some application of multiple correlation methods. Fortunately the methods of multiple curvilinear correlation, as presented in Chapters 14, 15, and 16, can be extended to treat non-quantitative factors as well, and thus provide the answer to the difficulty.

Eliminating the Influence of Other Variables.—The method of determining regressions for non-quantitative variables may be illustrated by the data shown in Table 70. These data are from a study of the relation of various quality factors to the price of eggs sold at retail.[1] The factors shown in Table 70 are X_2, an index of the interior quality of the

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[1] Original data collected by C. B. Howe. See reference 42 of Chapter 22.

TABLE 70

DATA FOR EGG PROBLEM, WITH A NON-QUANTITATIVE INDEPENDENT VARIABLE

Independent variables				Dependent variable, X_1	z'''	$f(X_5)$	z''''
X_2	X_3	X_4	X_5 *				
21	23	4	C	35	− 7.3	+0.6	− 7.9
35	24	12	C	45	− 8.4	+0.6	− 9.0
26	23	12	B	55	3.4	+0.9	2.5
27	24	12	B	55	3.3	+0.9	2.4
31	22	12	A	50	− 1.8	−2.5	0.7
35	24	12	C	44	− 9.4	+0.6	−10.0
28	23	12	C	60	8.2	+0.6	7.6
41	23	12	B	50	− 4.8	+0.9	− 5.7
28	26	2	C	45	− 1.6	+0.6	− 2.2
24	23	11	B	52	4.6	+0.9	3.7
28	20	12	C	45	− 5.5	+0.6	− 6.1
49	24	12	C	55	− 3.6	+0.6	− 4.2
30	24	12	C	55	2.4	+0.6	1.8
48	23	12	B	60	1.9	+0.9	1.0
19	22	9	C	45	1.8	+0.6	1.2
22	23	3	A	45	1.7	−2.5	4.2
33	25	12	C	60	6.6	+0.6	6.0
26	24	12	C	59	6.9	+0.6	6.3
35	23	12	B	55	2.1	+0.9	1.2
20	23	12	B	50	− 0.9	+0.9	− 1.8
25	25	12	B	55	2.6	+0.9	1.7
46	24	12	B	60	2.5	+0.9	1.6
30	26	1	B	45	− 3.2	+0.9	− 4.1
24	24	12	B	55	3.1	+0.9	2.2
48	23	12	B	60	1.9	+0.9	1.0
17	22	12	C	55	4.8	+0.6	4.2
18	22	12	A	45	− 5.3	−2.5	− 2.8
41	24	12	C	55	− 0.3	+0.6	− 0.9
30	25	12	C	67	14.0	+0.6	13.4
19	24	2	B	53	8.3	+0.9	7.4
47	24	0	B	55	0.9	+0.9	0.0
32	24	12	B	55	2.2	+0.9	1.3
26	24	12	B	49	− 3.1	+0.9	− 4.0
38	24	12	A	42	−12.2	−2.5	− 9.7
29	23	12	B	42	− 9.9	+0.9	−10.8
24	24	0	A	45	− 2.9	−2.5	− 0.4
37	25	12	A	40	−14.3	−2.5	−11.8
36	23	12	A	48	− 5.1	−2.5	− 2.6

* A designates "sold without carton," B "sold in carton but unbranded," and C "sold in carton with brand name."

see fig z''' p. 245

TABLE 70—*Continued*

Independent variables				Dependent variable, X_1	z'''	$f(X_5)$	z''''
X_2	X_3	X_4	X_5 *				
10	23	0	B	47	1.2	+0.9	0.3
35	24	12	C	59	5.6	+0.6	5.0
22	22	12	B	52	1.2	+0.9	0.3
29	21	12	B	55	4.0	+0.9	3.1
16	23	0	B	40	−6.5	+0.9	−7.4
6	22	3	B	40	−1.0	+0.9	−1.9
31	23	12	B	55	2.8	+0.9	1.9
26	23	12	B	55	3.4	+0.9	2.5
36	21	12	B	60	7.8	+0.9	6.9
39	22	12	B	55	1.4	+0.9	0.5
42	23	12	B	60	4.8	+0.9	3.9
36	24	12	C	60	6.4	+0.6	5.8
47	22	12	B	60	2.8	+0.9	1.9
27	24	12	C	55	2.8	+0.6	2.2
31	22	12	A	50	−1.8	−2.5	0.7
26	22	11	A	40	−7.2	−2.5	−4.7
45	23	12	A	60	3.5	−2.5	6.0
18	25	12	C	45	−6.6	+0.6	−7.2
35	24	12	C	50	−3.4	+0.6	−4.0
21	23	12	C	55	4.0	+0.6	3.4
44	23	12	A	60	3.9	−2.5	6.4
48	24	12	A	55	−3.6	−2.5	−1.1
33	24	12	A	55	2.0	−2.5	4.5
47	24	12	C	55	−3.1	+0.6	−3.7
16	22	5	A	45	3.9	−2.5	6.4
32	25	0	B	50	0.8	+0.9	−0.1
45	25	12	B	55	−2.4	+0.9	−3.3
46	23	12	B	57	0.0	+0.9	−0.9
32	24	12	C	55	2.2	+0.6	1.6
16	23	1	C	41	−4.2	+0.6	−4.8
30	25	1	C	50	2.3	+0.6	1.7
24	22	0	A	42	−5.0	−2.5	−2.5
44	24	11	B	50	−2.6	+0.9	−3.5
25	22	12	B	49	−2.1	+0.9	−3.0
16	23	0	A	45	−1.5	−2.5	1.0
31	24	8	A	48	3.2	−2.5	5.7

* A designates "sold without carton," B "sold in carton but unbranded," and C "sold in carton with brand name."

eggs in each dozen; X_3, the weight of each dozen in ounces; X_4, the number of white eggs in each dozen; X_5, the type of carton the eggs were sold in; and X_1, the price of eggs per dozen, in cents. Net curvilinear regressions have been determined for the three quantitative factors by the successive approximation method, and estimated prices have been worked out by the regression equation

$$X'_1 = a' + f(X_2) + f(X_3) + f(X_4)$$

The residuals, z''', obtained by subtracting these estimated prices from the observed prices, X_1, are shown in the table. The values in the last two columns are explained later.

Determining the Net Influence of the New Variable.—The first step in determining the net regression of X_1 on X_5 is to group the residuals from the previous curves, z''', according to the new factor X_5, and determine the average for each group. This gives results as follows:

Value of X_5	Average of z'''
A—no carton	−2.5
B—carton	+0.9
C—carton and brand name	+0.6

$f(X_5)$

These results show that, after making allowances for the size, color, and quality of the eggs, those with unmarked cartons sold 3.4 cents above those sold in bulk, on the average, but those with branded cartons sold only 3.1 cents above eggs in bulk. These results cannot be accepted as the final effect of package on price without first raising the question whether the curves previously determined to show the influence of the other factors might be changed somewhat were the type of package taken into account. Whether this will be true or not depends upon whether there is any correlation between the new factor and the factors previously considered, or whether they are quite independent of each other. This can be determined by sorting the other factors according to the values of X_5, and determining their averages for each group. The results are as follows:

Value of X_5	Averages of other independent variables			Number of cases
	X_2	X_3	X_4	
A—no carton	30.6	23.1	8.6	17
B—carton	31.6	23.2	9.6	33
C—carton and brand	29.9	23.8	10.2	24

It is evident that there *is* some correlation between X_5 and the other variables. Apparently the eggs sold in unmarked cartons are, on the average, of the best quality and of medium size; the eggs sold in cartons under brand names are of larger size, but are not of such high quality, on the average, whereas those sold in bulk average medium in quality but low in size.[2] Accordingly, the curves previously determined for the change in price with differences in size and

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[2] The exact correlation between X_5 and X_2, X_3, and X_4 can be computed by estimating each of the other variables from the values of X_4, using the averages of X_2, X_3, and X_4 for each group of X_5 as the estimated values of X_2, X_3, and X_4, for the cases falling in each group. The residuals between the estimated and actual values, and their standard deviation, can then be computed for each of the three variables. Then the indexes of correlation can be computed in the usual way. When computed this way by using group averages instead of a continuous function, the special name *correlation ratio* is given to the correlation, and the symbol η is used to designate it. This value may be more rapidly computed by the following formula (using Y to represent the dependent variable, and X the independent variable, just as with simple correlation in Chapters 5 to 7):

$$\eta_{yx} = \sqrt{\frac{\Sigma[n_0(M_0)^2] - n(M_y)^2}{n\sigma_y^2}} \qquad (68)$$

Here η_{yx} is the correlation ratio for Y values estimated from group averages when sorted on X; $n_0(M_0)^2$ is the number of cases in each group times the square of the average value of Y for that group, $\Sigma[n_0(M_0)^2]$ is the sum of all such values, σ_y is the standard deviation of the variable being estimated, and n is the number of all the observations $(= \Sigma n_0)$.

The process may be illustrated by calculating η_{25}, the correlation ratio between X_2 and X_5, from the data above:

X_5	M_0 Mean X_2	n_0 Number of cases	$(M_0)^2 n_0$
A	30.6	17	15,918.12
B	31.6	33	32,952.48
C	29.9	24	21,456.24
Σ	....	74	70,326.84

$$\eta^2_{25} = \frac{\Sigma[n_0(M_0)^2] - n(M_2)^2}{n\sigma_2^2} = \frac{70{,}326.84 - 70{,}285.61}{7{,}505.76} = .005493, \quad \eta_{25} = 0.074$$

The value as calculated is subject to the same correction equation (26) as the correlation index, with m = number of groups.

So adjusted, η_{25} shrinks to 0, showing no real correlation.

This same measure of correlation by group averages can be applied to quantitative variables as well as to non-quantitative ones, but in that case it has less significance than the index of correlation, which relates to a continuous function instead of an irregular line of averages.

in quality may have included some portion of the effect really associated with cartons instead. Now that at least an approximate measure has been obtained of the influence of carton on price, the previous curves may be modified by taking this factor also into account.

Taking Account of the Non-quantitative Variable in Estimating X_1 and z.—The first steps in the procedure of allowing for the extent to which prices varied with the carton are shown in Table 70. In the column headed $f(X_5)$ the approximate influence of differences in carton on price are entered, using the averages found in the tabulation on page 245. Since these values would be added to the previous estimated values of X_1 to obtain the new estimates, they may instead be subtracted from the previous residuals (z''') to obtain the revised residuals. The last column shows these new values for z''''. Before using these new values to see if any changes are necessary in the other regression curves we may first determine how much the standard error of estimate has been reduced by taking X_5 into account. This could be determined directly by computing the standard deviation of the new z'''' values; but a much shorter method is available, using the same principle employed in footnote 2. By the use of this method, the $\sigma_{z''''}$ may be computed from the $\sigma_{z'''}$ by the formula

$$\sigma^2_{z''''} = \frac{n\sigma^2_{z'''} - [\Sigma(n_0 M_0{}^2) - n(M_{z'''})^2]}{n}$$

The necessary computations are as follows:

X_5	$M_{z'''}$	Number of case	nM_0	$n(M_0)^2$
A	−2.5	17	−42.5	106.25
B	0.9	33	29.7	26.73
C	0.6	24	14.4	8.64
		Sums........	1.6	141.62

$$M_{z'''} = \frac{1.6}{74} = 0.0216$$

So

$$\sigma^2_{z''''} = \frac{74(5.06)^2 - (141.62 - 0.04)}{74} = 23.69$$

$$\sigma_{z''''} = 4.87$$

Computing the standard error for estimates based on X_5 and the other variables, we must recognize that the value of m has been increased by three by the introduction of the new factor; so while m was assumed to equal 8 previously, it now equals 11. Adjusting the values of 5.06 for $\sigma_{z'''}$ and 4.87 for $\sigma_{z''''}$ by equation (65), we find $\bar{S}_{1.f(2,\,3,\,4)} = 5.36$, and $\bar{S}_{1.f(2,\,3,\,4,\,5)} = 5.27$. Apparently the introduction of X_5 as a factor has had as yet but slight effect on the accuracy with which egg prices might be estimated.

Making Further Successive Approximation Corrections.—It is still possible, however, that the regressions for the other factors might be modified now that X_5 has been at least approximately allowed for. Accordingly the values of z'''' are classified according to the values of X_2, X_3, and X_4, and the averages computed for each group. The averages given in Tables 71, 72, and 73 are secured. The averages in Table 71 suggest that the curve for $f(X_2)$ might be modified slightly, so as to rise more steeply in the portion up to $X_2 = 40$, and less steeply thereafter. Table 72 does not indicate any significant relation between X_3 and z'''', so no further change in $f(X_3)$ is indicated. Table 73 indicates that the curve for $f(X_4)$ might also be altered slightly, so as to have a somewhat steeper slope.

TABLE 71

AVERAGE VALUES OF z'''' FOR CORRESPONDING X_2 VALUES

X_2 values	Number of cases	Average of X_2	Average of z''''
0–14	2	8.0	−0.9
15–19	9	17.2	−0.2
20–29	23	25.1	+0.1
30–39	24	33.5	+0.3
40–49	16	45.5	−0.1

TABLE 72

AVERAGE VALUES OF z'''' FOR CORRESPONDING X_3 VALUES

X_3 values	Number of cases	Average of z''''
20	1	−6.1
21	2	5.0
22	13	0.1
23	23	0.2
24	25	−0.1
25	8	0
26	2	−3.2

TABLE 73

AVERAGE VALUES OF z'''' FOR CORRESPONDING X_4 VALUES

X_4 values	Number of cases	Average of X_4	Average of z''''
0	7	0	−1.3
1− 2	5	1.4	−0.4
3− 5	4	3.8	+0.2
8−11	5	10.0	+0.5
12	53	12.0	+0.2

If $f(X_2)$ and $f(X_4)$ were modified as suggested, a new estimated value of X_1 might then be worked out, using these new curves and the previous curve for $f(X_3)$, and using the values for $f(X_5)$ already entered in Table 70. The new z's based on these new estimates might then be classified with respect to X_5, to determine if any change need be made in the values for $f(X_5)$ worked out on page 245. If any material change were found necessary in X_5, the residuals might be corrected accordingly, and then averaged with respect to X_2, X_3, and X_4, to see if any further changes would be needed in their values. This process of successive approximation should be continued until no further significant change was indicated in any of the curves, or until the $\bar{S}_{1.f(2,3,4,5)}$ showed no further reduction.

In view of the fact that none of the averages of z'''' shown in Tables 71 to 73 are so large but what they might very readily have occurred by chance, it does not seem worth while, in this problem, to carry out the additional steps just outlined. In a problem where the non-quantitative factor is an important one, however, and where it is significantly correlated with the other independent variables, the determination of the net function for that factor should be carried through a sufficient number of approximations to measure the final net effect of each factor as accurately as possible.

Taking the preliminary results shown on page 245 as the final measure of the influence of type of container on price, we may then conclude that eggs sold in an unmarked carton brought, on the average, 3.4 cents more per dozen than eggs of the same quality, size, and color sold in bulk, and 0.3 cents more than eggs sold in a carton with a brand name. (This last result might reflect the experience of consumers with branded eggs of poor quality, as indicated in the tabulation on page 245, which might tend to make them sell at a discount even when they were of equal quality.) The significance of the relation may be measured by the slight reduction in the standard error of estimate previously noted, or else by the increase in the index of multiple correlation. Computing the indexes of multiple correlation corresponding to the standard errors of estimate before and after the type of carton is allowed for, by equation (66), we find them to be $\bar{P}_{1.234} = 0.59$; $\bar{P}_{1.2345} = 0.62$. The corresponding indexes of determination, 35 and 38 per cent, indicate that taking into consideration the differences in the carton has increased the proportion of egg prices which can be explained by 3 per cent of the original variance, even after due allowance is made for the additional constants the process introduces into the estimating equation.

It should be noted that the first approximation to the regression on non-quantitative factors can be made directly from the first set of residuals, computed from the linear multiple regression equation, instead of waiting until after approximate regression curves are determined for the other factors. In case a non-quantitative factor is a very important one, so that ignoring it in determining the net linear regressions may seriously impair their accuracy, it may be roughly included by designating successive groups by a numerical code which approximates the expected influence of the variable. Then if the true influence is of a different order than the expected influence, that fact will show up when the first approximation curves are worked out. (For the non-quantitative factor the averages of residuals must be interpreted as discrete points for each class, however, rather than as a continuous function.) Thus for the egg problem it might have been tentatively assumed that eggs in branded cartons would sell above eggs in unbranded cartons, and both would sell well above eggs in bulk. The bulk eggs could then have been designated by 1; the unbranded cartons by 3; and branded cartons by 4. The net linear regression would have been positive; but the analysis of the residuals would have revealed that the eggs in branded cartons really averaged lower in price (other factors equal) than the eggs in unbranded cartons, so the final conclusion would probably be much the same as the one just determined.

Summary.—Where an independent factor is not a continuous variable, but may be classified into two or more groups, the regression of a dependent factor may be determined with respect to each group, while holding other factors constant by the usual multiple correlation process. Standard errors and indexes of correlation may be worked out to include the effects of non-quantitative independent factors equally as well as for continuously variable factors.

CHAPTER 18

DETERMINING THE RELIABILITY OF CORRELATION CONCLUSIONS

Early in this book it was pointed out that when any statistical constant, such as an average, is determined by means of a sample selected from a universe under study, the true value of that constant for all the cases in the universe might be quite different from the value determined for the sample. Methods were discussed which enable one to estimate within what range of the true average the mean of any one sample is likely to lie. Such estimates of the probable range enable one to judge how much confidence may be placed in a particular average, and with what degree of exactness it may be interpreted.

Simple Correlation.—*Regression Coefficients.*—Correlation constants determined from finite samples are just as subject to variation as are other statistical constants. Thus in an experiment when 5 samples of 30 observations each were drawn at random from the same universe, and the regression of Y on X was determined separately for each sample, the values for b_{yx} which were secured varied from -0.136 to $+0.449$, as shown in Table 74. When 5 samples of 50 observations each were drawn, and the regressions computed, the range was reduced to from

TABLE 74

VALUES OF b_{yx} SECURED IN SUCCESSIVE SAMPLES DRAWN FROM THE SAME UNIVERSE, WITH DIFFERENT NUMBERS OF OBSERVATIONS

	30 observations	50 observations	100 observations
	0.292	0.175	0.113
	0.012	−0.297	0.120
	−0.136	0.144	0.303
	−0.022	0.130	0.197
	0.449	0.167	0.132
True value...	0.152	0.152	0.152

-0.297 to $+0.175$; but the variation between samples was still large. Even when 100 observations were included in each sample, the regressions were by no means identical, though the range was reduced still more.

It is evident that the observed values of b_{yx} fell both above and below the true value for the universe from which the samples were being selected. It is also evident that the smaller the number of observations, the larger the variation in the results between different samples and the greater the possibility of a serious difference between the true value and that indicated by the sample. The amount of variation likely to be present in regressions determined from samples of any specified size may be estimated by the equation

$$\text{Standard error of } b_{yx} = \frac{\bar{S}_{y.x}}{\sigma_x\sqrt{n}} \qquad (69)$$

If only one of the samples in Table 74 had been obtained—say the first one with 50 observations—the observed value for b_{yx} would have been $+0.175$. The standard error of estimate for this sample was 2.46, and the σ_x was 2.44. Computing the standard error of b_{yx} for this sample by means of equation (69),

$$\sigma_b = \frac{2.46}{2.44\sqrt{50}} = \frac{2.46}{17.25} = 0.143$$

The value of b_{yx}, as determined from this single sample, may therefore be stated to be 0.175 ± 0.143. This simply means that the chances are two out of three (0.683) that the true value of b_{yx} lies between 0.319 and 0.032, and nineteen out of twenty (0.954) that it lies between 0.461 and -0.111. Of the five different samples, four of the coefficients actually lie within the smaller range stated, and four within the larger. The true value is also within the smaller range. This will not always be true, however. For example, if the sample had happened to give the same results as the third sample of 30 observations, with $b_{yx} = -0.136$, the case might have been different. For that sample, the values of the other constants were such as to make $\sigma_b = 0.109$. The value of b_{yx} as indicated by this sample, therefore, -0.136 ± 0.109, is such that the observed value lies 2.6 times its own standard error from the true sample. Although a departure as large as this would ordinarily be expected to occur only once in about 100 times (0.009), still it *may* happen with any particular sample. For that reason, if very great accuracy is desired, a range of three times the standard error may be used as the criterion, for there is but one chance out of nearly 400 (0.0027) that a constant such as a regression coefficient will fall more than three times its own standard error away from the true value for the universe.

These probabilities apply only in case there are thirty or more observations in the sample. As was pointed out in Chapter 2, if the number of observations is less than thirty, the probabilities of falling outside of any given range of the true value are increased, as shown in Table A. In using this table for regression coefficients, subtract 1 from the number of cases in the sample before looking the probability up in the table.[1]

Thus if a value of $b_{yx} = 0.50 \pm 12$ were found from a sample of 11 cases, the reliability of the observed regression could be judged from the column headed "10" in Table A. That indicates that there are 34 chances out of 100 that the true regression may be smaller than 0.38 or larger than 0.62 (a range of 1 standard error); 7.7 chances out of 100 that it is smaller than 0.26 or larger than 0.74 (a range of 2 standard errors); and $1\frac{1}{2}$ chances out of 100 that it is smaller than 0.14 or larger than 0.86. The odds are thus better than 98 to 2 that the true value is between 0.14 and 0.86—if the sample was drawn under such conditions that the formulas of simple sampling hold true.

It should be noted from equation (69) that the standard error of the regression coefficient varies inversely with the square root of the number of observations. The effect of this is illustrated in Table 74. The variation of the regression coefficients obtained from samples of 100 observations is only about half as great as the variation of the regression coefficients from samples of 30.

Not only may the observed *slope* of the regression line vary from the true slope, but the elevation of the line, as observed from a sample, may vary from the true elevation. Formula (69) has already indicated a way of determining the standard error of the regression coefficient, and so of estimating the probable range within which the true slope lies. The height of the regression line is most accurately determined for the mean estimated value, $M_{y'}$, of the dependent factor, corresponding to the observed mean value of X, the independent factor. Defining that mean

$$M_{y'} = a_{yx} + b_{yx}M_x$$

its standard error may be found by the formula

$$\sigma_{M_{y}'} = \frac{\bar{S}_{y.x}}{\sqrt{n}} \tag{70}$$

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[1] That is because two constants (a and b) have been determined simultaneously in the process of getting b, whereas the table is stated for arithmetic means, which represent the determination of only a single constant.

The standard error of the whole regression line may now be determined from equations (69) and (70). We may illustrate by data from the cotton yield problem used as an example in Chapter 6. With 14 observations, the values were $b_{yx} = 16.70$, $a_{yx} = -2.261$, $M_x = 1.97$, $\bar{S}_{yx} = 8.28$, $\sigma_x = 0.73$

$$M_{y'} = -2.261 + (16.70)(1.97) = 30.64$$

$$\sigma_{M_{y'}} = \frac{8.28}{\sqrt{14}} = 2.21$$

$$\sigma_{b_{yx}} = \frac{8.28}{0.73\sqrt{14}} = 3.03$$

Since the estimated value, Y' equals $M_{y'} + b(x)$, the standard error of the estimate for any value of x will be composed of the sum of the standard errors of $M_{y'}$ and of $b(x)$. Standard errors are standard deviations, hence they can be summed only by adding their squares (as demonstrated in Technical Note 1). The standard error of Y', for any particular value of x, is therefore given by the equation [2]

$$e_{y'} = \sqrt{\sigma^2_{My'} + (\sigma_{b_{yx}} x)^2}$$

Using this relation, the calculation of the standard error of Y', for selected values of X, is shown in the following tabulation:

Selected values of X	Departures from mean x	Calculation of $\sigma_{y'}$				
		$\sigma_{b_{yx}}x$ $= (3.03x)$	$(\sigma_{b_{yx}}x)^2$	$\sigma^2_{My'}$ $= (2.21)^2$	$\sigma^2_{y'}$ $[(\sigma_b x)^2 + \sigma^2_{My'}]$	$\sigma_{y'}$
0.97	−1.00	−3.030	9.1809	4.8841	14.0650	3.75
1.47	−0.50	−1.515	2.2952	4.8841	7.1793	2.68
1.97	0	0	0	4.8841	4.8841	2.21
2.47	0.50	1.515	2.2952	4.8841	7.1793	2.68
2.97	1.00	3.030	9.1809	4.8841	14.0650	3.75
3.47	1.50	4.545	20.6570	4.8841	25.5411	5.05
3.97	2.00	6.060	36.7236	4.8841	41.6077	6.45

There are 14 cases; subtracting the one extra constant involved in correlation determinations gives 13 as the number of observations with which to judge from Table A the significance of these standard errors. Taking values midway between those for 10 and for 16 cases, we find the odds are 33.8 out of 100 that the true values of b_{yx} and of $M_{y'}$ do not differ from the observed values by more than the calculated standard errors; or 7 out of 100 that they do not differ by more than twice the calculated standard errors. The chances are therefore 93 out of 100 that the true regression line would fall within twice the standard errors just calculated. Plotting $2\sigma_{y'}$ above and below the corresponding values of Y', given by the regression line, shows this range. These limits are plotted in Figure 52, together with the original observations and the regression line. The limits within which the line probably fell could be shown in a similar manner for any other desired limit of probability. It is now clear why great caution must be exercised in extending even a linear regression line beyond the range of the data from which it is derived. As is evident in the figure, the true position of the line becomes very uncertain as the limits of the data are approached, and increases rapidly beyond them.

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[2] Working, Holbrook, and Hotelling, Howard, Applications of the theory of error to the interpretation of trends, Jour. Amer. Stat. Assoc., Papers and Proc., xxiv, pp. 73–85. March supplement, 1929.

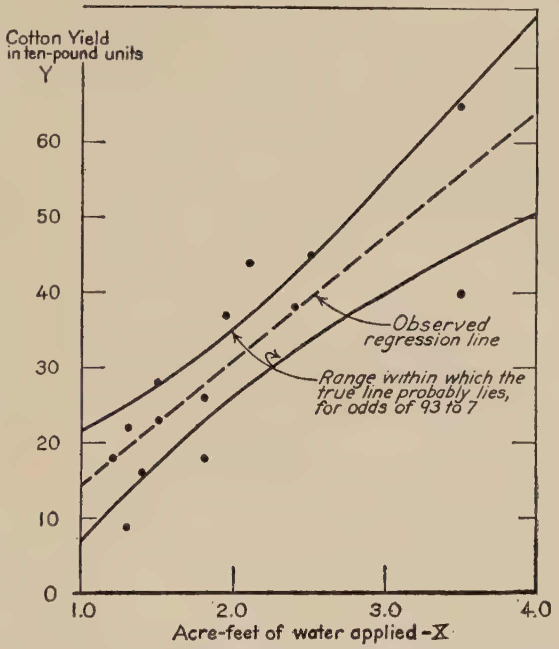

FIG. 52.—Linear regression of cotton yield on irrigation water applied, and range within which the true relation probably lies.

Correlation Coefficients.—In exactly the same way that regression coefficients will vary from sample to sample, all other statistical constants tend to vary. Regression coefficients tend to be normally distributed around the true value, so that the probability of a given departure from the true value occurring may be judged from the normal curve; but that is not equally true of correlation coefficients. If the number of observations in the sample is exceedingly large, so that fairly stable results are secured, the distribution of the observed correlations will tend to be fairly normal, so that the standard error may be estimated by the formula

$$\text{Standard error of } r_{yx} = \frac{1 - r^2}{\sqrt{n - 2}} \tag{71}$$

This equation applies only when n is large. To test the significance of correlation coefficients obtained from small samples, Fisher has developed the equation

= No Times STD. ERROR.

$$t = \frac{r\sqrt{n - 2}}{\sqrt{1 - r^2}}$$

The value t is used to judge the probability of the occurrence of such a correlation purely by chance, in exactly the same way that the number of times an average is times its standard error is used to judge the probability of the significance of the average. Thus if a correlation of 0.60 is secured with a sample of 21 cases, $t = 3.26$. Looking up this value in Table A or Figure *A*, using 20 for n,[3] we find that there is less than 1 chance out of 200 that such a value could have occurred by chance. If, however, a correlation of 0.60 had been secured with only 7 cases, t would equal 1.68. Figure *A* indicates that with this value of t, there would be almost 16 chances out of 100 that the observed correlation might have been obtained by chance from a universe in which the true correlation was zero.[4]

But while this method may be used in conjunction with Table A to determine whether or not the correlations computed from small samples are any valid indication of a correlation in excess of zero, it cannot be used to determine the significance of the difference in correlation between two samples, or to determine whether or not the correlation in a given sample exceeds any specific value. In the first illustration, for example, where $r = +0.60$, one might wish to know the probability that the true correlation in the universe exceeds $+0.20$. Owing to the skewed distribution of values of r when computed from small samples, this cannot be determined by a simple sampling formula. R. A. Fisher has devised a method, however, of so transforming observed values of r as to give them a normal distribution, and then solving such problems as this from the transformed values.

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[3] Just as with regression coefficients, 1 less than the number of cases should be taken for n when Table A is used to judge the significance of a correlation coefficient. The unadjusted correlation, r, should be used in all tests of significance; *not* the adjusted value $\bar{r}$.

[4] See R. A. Fisher, Statistical methods for research workers, pages 159 to 162, for a fuller discussion of the use of t in judging the reliability of correlation coefficients.

For methods of dealing with this phase of sampling, the reader is referred to his presentation of the method in Statistical Methods for Research Workers, 2nd ed., pages 163 to 173.

Certain of Fisher's methods to determine the reliability of observed correlations may be put into more simple form for general use, as shown in Figure *B* at the end of this book. This figure is based upon the idea that, if the chances are 19 to 1 that the true correlation is at least a specified value, that value will be a reasonable one to use as the probable minimum correlation existing in the universe from which the sample was obtained. These probable minimum values, for odds of 19 to 1, have been worked out by Fisher's methods, and the results plotted for different sizes of sample and observed correlations. Thus if a sample of 20 gives an observed correlation of 0.70, the figure shows at a glance that the chances are 19 out of 20 that the true correlation is 0.44. Similarly, for an observed correlation of 0.55 with a sample of 35 cases, reading from the line for observed correlation = 0.55, and interpolating between $n = 30$ and $n = 40$, gives 0.32 as the correlation which probably exists (for 0.95 probability). In a similar manner the figure can be used for any other size of sample up to 100, and any observed correlation, to determine the minimum correlation for odds of 0.95.

Correlation Indexes.—The reliability of indexes of (curvilinear) correlation, ρ, determined from very large samples, may be judged by the use of the following equation:

$$\text{Standard error of index of correlation} = \frac{1 - \rho^2}{\sqrt{n - m}} \qquad (72)$$

In using Table A to test the significance of such correlations for small samples by the t method, 1 less than the number of constants necessary to represent the regression line mathematically [the value m of equation (26), minus 1] must be deducted from the number of cases before using Table A. Thus if a correlation index computed for a cubic parabola fitted to 7 observations were to be judged, its reliability would be determined by using the column headed 4. Since 4 constants would be represented in the regression equation, $7 - (m - 1) = 4$. If the computation gave $t = 2.8$, there would be 7 chances out of 100 that the correlation really did not exceed zero.

Indexes of correlation probably tend to be skewed in their distribution, just as do coefficients of correlation; therefore parallel special methods must be employed in judging their significance. Figures *C*, *D*, and *E*, prepared to apply to multiple correlation coefficients in the same way that Figure *B* applies to simple correlation coefficients, may be used tentatively to judge the reliability of indexes of correlation, until more exact measures have been developed. Where $m = 4$, Figure C (for $R_{1.234}$) may be used; where $m = 6$, Figure D (for $R_{1.23456}$), and where $m = 8$, Figure E (for $R_{1.2345678}$). These figures, also, are based upon methods developed by R. A. Fisher.

Multiple Correlation

Coefficients of Multiple Correlation and Net Regression.—Correlation constants derved from multiple regression studies are even more subject to chance variation than are those from simpler analyses. In a sample of 30 cases drawn from a known universe, for example, the following values were obtained:

$$R_{1.234} = 0.538;\ b_{12.34} = 0.583;\ b_{13.24} = 0.366;\ b_{14.23} = 0.949$$

Drawing 15 more samples of 30 cases each, 10 of 50 cases, and 5 of 100 cases, values were secured for R and the b's as shown in the following statement and in that on the next page.

Distribution of Values for Multiple Correlation Coefficients for Repeated Samples Drawn from the Same Universe (True Value 0.563)

Range of values	30 observations	50 observations	100 observations
0.300–0.399	4	1	
0.400–0.499	3	5	
0.500–0.599	4	1	4
0.600–0.699	4	3	1
0.700–0.799	1		

From these two tables we can see how the variation decreases as the number of cases increase, and can also see what values the constants from the several samples tend to center around, and so estimate the approximate true value. But some definite idea of the range within which this true value probably would lie could have been obtained by computing the standard error of each constant by means of the formulas:

$$\left.\begin{array}{l}\text{Standard error for a coefficient of}\\ \quad\text{multiple correlation } R_{1.234\ldots}\end{array}\right\} = \frac{1 - R^2_{1.234\ldots}}{\sqrt{n - m}} \tag{73}$$

$$\left.\begin{array}{l}\text{Standard error for a coefficient}\\ \quad\text{of partial regression } b_{12.34\ldots}\end{array}\right\} = \sqrt{\frac{\bar{S}^2_{1.234\ldots}}{n\sigma_2{}^2(1 - R^2_{2.34\ldots n})}} \tag{74}$$

The standard error for the value of R, 0.538, given for our first sample works out to be 0.139. If we ignore the fact that the distribution of R, Distribution of Values for Net Regression Coefficients for Repeated Samples Drawn from the Same Universe

Range of values	30 observations	50 observations	100 observations	True value
Values for $b_{12.34}$:				
−0.79 to −0.60	1			
−0.59 to −0.40	0			
−0.39 to −0.20	1			
−0.19 to −0.00	0	2		
0 to 0.19	2	1	1	
0.20 to 0.39	6	4	4	+0.320
0.40 to 0.59	4	3		
0.60 to 0.79	2			
Values for $b_{13.24}$:				
−0.19 to − 0	2			
0 to 0.19	3			
0.20 to 0.39	5	6	2	+0.377
0.40 to 0.59	2	2	2	
0.60 to 0.79	1	1	1	
0.80 to 0.99	2	1		
1.00 to 1.10	1			
Values for $b_{14.23}$:				
0 to 0.19				
0.20 to 0.39	....	1		
0.40 to 0.59	1	1		
0.60 to 0.79	8	2	2	
0.80 to 0.99	3	4	3	+0.824
1.00 to 1.19	4	1		
1.20 to 1.39	....	1		
1.40 to 1.59				

just as of r, is not normal, we may interpret that very roughly by saying that the chances are two out of three that the true value of R lies between 0.40 and 0.68. As it happens for this particular sample, the true value, 0.563, does lie within this range. Ten of the 16 samples, or 63 per cent, gave values falling within 0.139 of the true value, so the computed standard error is not so misleading in this particular case. For still smaller samples, or for higher correlations, the standard error computed by equation (73) would be less reliable. For those cases, Table A may be used to judge whether there is real evidence that the true correlation exceeds zero, just as in the case of the correlation coefficient and index. In using that table, $m - 1$ must be subtracted from the number of observations. (This also applies in using Table A for coefficients of net regression.) For more exact interpretations, Fisher's transformation method, previously referred to, may be utilized.[5]

For small samples, the reliability of coefficients of multiple correlation varies not only with the correlation and the size of sample, but also with the number of independent variables. Fisher has recently[6] worked out an exact method for judging the probable significance of observed coefficients of multiple correlation. Figures *C*, *D*, and *E* provide a simple method of applying his conclusions for multiple correlation coefficients, in the same way that Figure *B* provides for simple correlation coefficients. For problems involving 3, 5, and 7 independent factors, respectively, these figures show the minimum true correlation that probably exists in the universe, for a probability of 0.95, for with any size of sample up to 100, and for any observed correlation. Thus if with 30 observations, a correlation of $R_{1.23456} = 0.80$ should be obtained, the odds are 19 out of 20 that the true correlation (from Figure *D*) is at least 0.58. Similarly, if for 50 observations a correlation of $R_{1.234} = 0.62$ were obtained, Figure *C* gives 0.42 as the probable minimum correlation in the sample. These conclusions, of course, apply only if the conditions of simple sampling are fulfilled. Problems with 2, 4, or 6 independent variables may be considered by interpolating between the corresponding values given for 1, 3, 5, or 7 independent variables.

Considering the problem mentioned above, where a sample of 30 observations showed $R_{1.234} = 0.538$, Figure *C* gives a value of 0.16 as the probable minimum correlation. From the single sample we could then say that the true correlation is probably at least 0.16 in the universe from which the sample was drawn.

Turning to the meaning of the regression coefficients, we may illustrate the case with one constant, $b_{12.34}$. The value given by the original sample was 0.538. For that sample $\sigma_2 = 2.53$, $\bar{S}_{1.234} = 2.81$, and $R_{2.34} = 0.708$. Substituting these values in equation (74) the standard error works out to be 0.287. The observed regression may therefore be stated to be 0.538 ± 0.287. This indicates that the chances are 2 out of 3 (0.683), that the true value lies between 0.251 and 0.825, and 19 out of 20 (0.955) that it lies between -0.036 and 1.112. Actually, the true value in this case was 0.320, or within the first range. It may be noted that 11 of the 16 samples showed regression coefficients for $b_{12.34}$ within 0.287 of the true value, and all but 1 fell within 0.574 of the true value. Again this illustrates how the variability of constants which tend to be normally distributed may be estimated by appropriate error formulas, and hence how the reliance to be placed in conclusions from a given sample may be judged.

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[5] See pages 366 to 369 in Appendix 1, Methods of computation, for the most effective method of computing the standard errors of net regression coefficients, according to equation (74).

[6] Fisher, R. A., The general sampling distribution of the multiple correlation coefficient, Proc. Royal Soc., A, Vol. 121, pp. 654–673, 1928.

From equation (74) it is evident that the reliability of a regression coefficient varies directly with the multiple correlation of the dependent factor with the other factors, but inversely with the multiple correlation of the particular independent factor with the other independents. The more closely a particular independent factor can be estimated from the other independent factors present, the less accurately can the net relation of the dependent factor to it be determined.

The qualifications the use of this error formula throws around regression results may be illustrated in a problem where the theory of sampling is fairly applicable, namely, the relation between the feed a herd of cows receives and the resulting milk production. Table 75 shows these results for two different studies.

This table illustrates two points: first, that the regression results are not very accurate even with a multiple correlation of 0.80; and second, that the reliability of the regressions varies from variable to variable, being much greater in some cases than in others. It is obvious that some of the regressions would have no statistical significance at all, whereas others would indicate the probable relations within a fairly close range of accuracy.

Thus for the per cent of lime, with the P.E. = 67.4 per cent of the regression, there is 1 chance out of 2 that the true net regression varies from that observed in this sample by 2/3 of the observed value, and 1 chance out of 6 that the true net regression is of opposite sign from that observed. With the total digestible nutrients, on the other hand, with the probable error only 12 per cent of the observed value, there is but little chance that the observed value differs from the true regression by more than 30 per cent, and very little chance that it differs as much as 40 per cent.

If the regression equation is to be used solely as a basis for making new estimates of the value of the dependent factor to be expected for given values of the independent factors, then it does not make such a great difference as to the accuracy of the several regression coefficients. Any deficiency in one will tend to be compensated for by an excess in another, and the whole set will provide estimates of the accuracy indicated by the standard error of estimate. But if, on the contrary, the major interest is not solely in the total estimate,

TABLE 75

PROBABLE ERRORS OF PARTIAL REGRESSION COEFFICIENTS, IN PER CENT OF THE VALUE OF THE COEFFICIENT *

Item	Wisconsin study	Minnesota study
Number of observations	95	77
Number of variables	10	8
Multiple correlation, adjusted for number of variables	0.805±0.039	0.862±0.034

PROBABLE ERROR OF REGRESSION COEFFICIENTS †

Independent variable	Per cent	Per cent
Total digestible nutrients	12.0	11.5
Nutritive ratio	12.4	9.5
Per cent of protein "good"	28.3	
Per cent of lime	67.4	
Per cent summer feeding	17.5	
Per cent silage	21.5	13.7
Fat test of milk	10.6	3.7
Per cent fall freshening	18.5	11.8
Value per cow	26.8	
Age of cows	....	17.9
Per cent grain in ration	....	20.2

* Ezekiel, Mordecai, The application of the theory of error to multiple and curvilinear correlation, Jour. Amer. Stat. Assoc., Vol. XXIV, No. 165 A, March, 1929, Supplement, p. 103.

† The coefficients are for the net regression of milk production on the factors stated. P.E. = 0.6745 of the standard error.

but in the changes in the dependent factor with changes in each particular independent factor, then the reliability of each particular regression coefficient becomes of very real importance. In the illustration cited, for example, it would not do to know merely that the milk production per cow varied both with protein content and with lime, if it was desired to know how much to allow for protein and how much for lime in compounding a ration. Instead, the probable errors indicate that the influence of protein has been fairly accurately measured, whereas the influence of lime has not been accurately measured at all. Not much confidence therefore can be placed in the conclusions as to this latter factor.

Multiple Curvilinear Correlation.—All of the formulas for multiple regression constants cited up to this point have been derived for linear correlation use. For curvilinear multiple correlation results, however, no measures of the probable error have yet been devised for the free-hand process by logical and mathematical deduction. Experiments mentioned previously were initiated to provide at least some empirical measures of reliability. The results indicate that the *index* of multiple correlation must be corrected for the number of constants involved or assumed just as much as the coefficient of multiple correlation, as has already been illustrated.

The reliance to be placed on regression curves requires separate treatment. Where those curves are determined by fitting mathematical functions, the probable accuracy with which the true relation is expressed by the mathematical curve may be judged by error formulas which have been worked out mathematically by an extension of the same methods upon which those previously presented were based. For regression curves determined by the successive approximation process or by the graphic approximation process, no such mathematical treatment is possible. The experimental study of the reliability of regression curves determined by successive approximations, however, has thrown some light on the reliability of such curves, and made it possible to state the following general principles:

First, the reliability of regression curves varies inversely with the standard error of estimate for the entire sample.

Second, the reliability of any point on a regression curve varies directly with the square root of the number of observations on which that portion of the curve was based.

Third, the reliability of any point on a regression curve, when stated as the difference between the value of the function at that point and the value of the function at the point corresponding to the mean of the independent variable, varies inversely with the square root of the distance the selected point is from the mean of the independent variable, measured in units of the standard deviation of the independent variable.

All of these points apply equally to simple regression curves and net regression curves, computed while holding the influence of other factors constant. For net regression curves, one further point is involved, the extent to which one independent factor tends to vary with the other independent factors, which may be stated:

Fourth, the reliability of points on a net (or partial) regression curve varies inversely with the multiple correlation of the particular independent factor with the other independent factors.

Of these four points, comparison with equation (74) shows that the first and fourth apply equally to linear net regressions; that the second applies also, but to the total number of cases instead of to the number along any portion of the curve, while the third point applies somewhat differently in the two cases, although present in both.

Technical note 13 gives some tentative formulas for the standard error of regression curves, which embody these results. The adequacy of these formulas has not yet been definitely established, however, and they are given only as first approximations subject to subsequent modification.

Regression Curves Fitted Mathematically.—Where regression curves are obtained by fitting definite mathematical equations to the data, the standard error of the curve may be judged by the same methods just presented for determining the probable errors of net regression coefficients. Thus if a parabola of the formula

$$X_1 = a + b(X_2) + b'(X_2^2)$$

is determined, the standard errors of b and b' may be determined by equation (74), treating X_2 and X_2^2 as two independent variables. The range within which the true curve probably lies may then be worked out just as has been illustrated for a linear regression. Similarly, if net regression curves are determined by fitting several mathematical equations simultaneously (as presented in detail in Chapter 21), this same method may be used to judge the reliability of each of the net regression curves so obtained.[7]

Summary.—Coefficients of correlation and of regression, indexes of correlation, and regression curves, when determined from a limited sample, may depart more or less widely from the true value for the universe from which that sample was drawn. This chapter presents methods by which the probable reliability of correlation results may be judged. These methods represent an extension of the methods presented in Chapter 2 for judging the reliability of averages.

The methods of this chapter apply only when the correlations are determined from samples so selected as to comply with all the assumptions of simple sampling. Where the samples are selected by other methods, the results may be of greater or of less reliability than if random sampling had been employed. Furthermore, in many types of problems, such as in time series, the observations can hardly be regarded as samples drawn from a universe. In such cases, statistical measures of reliability have but little meaning.

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[7] See Schultz, Henry, The standard error of a forecast from a curve, Jour. Amer. Stat. Assoc., pp. 139–185, June, 1930.

CHAPTER 19

INFLUENCE OF SELECTION OF SAMPLE AND ACCURACY OF OBSERVATIONS ON CORRELATION RESULTS

Selection of Sample.—Methods of determining linear and curvilinear regressions, together with appropriate measures of their significance and accuracy, have been set forth in previous chapters. These methods do not yield results representative of the universe from which the sample observations have been drawn, however, if that sample is not truly representative of the particular relation being determined. There are various ways in which the sample may fail to represent the universe, and the resulting extent to which the correlation constants will be biased will vary both with the character of the unrepresentativeness and with the individual coefficients. Each type of abnormality must therefore be treated separately.

The samples may be selected from the universe in such a way as to exclude all of the observations falling beyond a certain value of a given variable, thus ruling out values either at one or at both extremes, or perhaps ruling out middle values, and selecting only extreme ones. This may be done for either the dependent variable, or the independent variable or variables, or for both together. Such a selection of observations produces certain specific effects upon the correlation constants. Under some conditions it may be very desirable to select the observations in this way, if only the resulting aberrations in the correlation constants are recognized and allowed for.

A second and somewhat more difficult type of problem to deal with arises when there are errors of measurement in obtaining the values of one or more of the variables—such errors as might arise, for example, in estimating the total production of corn in the United States within a given year, or in working out, from a farmer's memory, what was the income on his farm the previous year. Here again the effect on the correlation constants will depend upon which variable or variables are affected by the errors, so a separate discussion must be given each case.

The clearest way of indicating the effect of these various departures from truly representative sampling may be by first stating the general principles involved, and then illustrating the way those principles work out by concrete illustrations. Except where specially stated otherwise, the discussion will apply solely to linear relations. The effects for curvilinear relations are in general analogous to those to be discussed.

Selection of Sample with Respect to Values of Independent Variable.—If the sample is selected with respect to values of the independent variable, that will not tend to affect the slope of the regression line, but will affect the value of the coefficient of correlation. If the selection is such that extreme values are rejected but intermediate values are left in, the correlation will be lowered below that prevailing in the universe; if intermediate values are rejected and only extreme values are used, the correlation will be raised above that prevailing in the universe. If the values of both variables are normally distributed, the standard error of estimate will tend to remain the same, regardless of the selection.

These principles may be illustrated by the set of hypothetical data shown in Table 76.

TABLE 76

Correlation Table, Showing Hypothetical Frequencies at Specified Values

Values of Y	Values of X				
	0	1	2	3	4
0	....	....	1	1	1
1	....	1	2	2	1
2	1	2	4	2	1
3	1	2	2	1	
4	1	1	1		

For the data shown, $\bar{r} = -0.47$, $\sigma_x = \sigma_y = 1.134$; and $b_{yx} = -0.50$. If now the values had been selected with reference to X, so as to exclude values below 1 or above 3, the number of observations would have been reduced from 28 to 22. For this restricted set of observations, $\bar{r} = -0.26$, $\sigma_x = 0.739$, $\sigma_y = 1.09$, but $b_{yx} = -0.50$. Computing the standard error of estimate, it comes out $\bar{S}_{y.x} = 1.02$ for the first case, and 1.07 for the second. It is quite apparent that the correlation has been lowered by the restriction in the selection of the values of X; but the regression of y on x has not been changed at all, while the standard deviation of the residuals has been only slightly changed (and that is because the distribution of the two variables is not exactly normal).

If now the selection be such that only extreme values of X are taken, say below 1 and above 3, the number of observations is reduced to 6. Computing the results for those values, $\sigma_x = 2.00$, $\sigma_y = 1.29$, $\bar{r} = -0.71$, but $b_{yx} = -0.50$! Also $\bar{S}_{y.x} = 1.00$.

Bringing the three sets of results together for comparison, we have the following:

With X used as independent variable:

	σ_x	σ_y	$\bar{r}_{xy}$	$\bar{S}_{yx}$	b_{yx}
All cases	1.13	1.13	−0.47	1.02	−0.50
Extreme values of X excluded	0.74	1.09	−0.26	1.07	−0.50
Only extreme values of X used	2.00	1.29	−0.71	1.00	−0.50

These three examples thus illustrate the principles stated before; that selection with respect to the independent factor does not tend to change the regression or the standard deviation of the residuals, but does affect the correlation, lowering the correlation if it has lowered the dispersion of the independent factor, and raising the correlation if it has increased the dispersion of that factor.

Selection of Sample with Respect to Values of Dependent Factor.—Selection with respect to values of the dependent factor is more serious, in that it affects all the constants. According as the effect is to raise or to lower the dispersion of the dependent factor, such selection tends to raise or lower both the regression coefficient and the coefficient of correlation from the value for the universe, and likewise to raise or lower, respectively, the standard error of estimate.

These principles may be illustrated from the three examples just used, by regarding X as the dependent factor and Y as the independent factor, and noting the influence of the selection with regard to the dependent factor, X, upon the regression of X on Y, b_{xy}. For the first case, with all values left in, $b_{xy} = -0.50$, and $\bar{S}_{x \cdot y} = 1.02$. For the second case, however, with extreme values of X left out, b_{xy} drops to -0.23, and $\bar{S}_{x \cdot y}$ becomes 0.73. For the third case, with only extreme values of X included, b_{xy} increases to -1.20, and $\bar{S}_{x \cdot y}$ becomes 1.55. Bringing these three sets together yields the following comparison:

With X used as dependent variable:

	σ_x	σ_y	r_{xy}	$\bar{S}_{xy}$	b_{xy}
All cases	1.13	1.13	−0.47	1.02	−0.50
Extreme values of X excluded	0.74	1.09	−0.26	0.73	−0.23
Only extreme values of X included	2.00	1.29	−0.71	1.55	−1.20

These results indicate the extent to which selection with regard to the dependent factor may completely destroy the significance of all the results.

Selection of Samples with Reference to Values of Both Variables.—Selection of cases with reference to values of both independent and dependent variables has even a greater effect upon the conclusions than the two cases discussed; selection of extreme values tending to exaggerate the correlation and regression, and of central values to lower both, to even greater extent than where the selection is with respect to the dependent factor alone.

If, in the data of Table 76, only those cases are selected which fall at 1 or below for both X and Y, and which have 3 or above for both X and Y, the observations are reduced to ten cases, as follows:

Values of X	Values of Y	Number of cases	Values of X	Values of Y	Number of cases
0	3	1	3	0	1
0	4	1	3	1	2
1	3	2	4	0	1
1	4	1	4	1	1

For these values, $\sigma_x = 1.48$, $\sigma_y = 1.48$, $\bar{r}_{xy} = -0.90$, $b_{yx} = -0.91$, and $\bar{S}_{yx} = 0.68$.

It is evident that such selection raises both the correlation and the regression above the true value for the universe. This is to be expected, for this selection is equivalent to picking out the pairs of values which *do* show correlation with each other. Restricting the selection to paired values of above 1 for both variables, and below 3 for both variables, likewise would be picking out cases so as to eliminate all correlation. Such selection obviously destroys the value of the results.

Conclusions with Reference to Selection of Data.—If an investigator is interested only in the regression line and not in the degree of correlation, and if the regression is truly linear, selection of data with reference to the independent factor (or factors) will not tend to change the slope of the regression line (or lines). Under those conditions selection of extreme cases of the independent factor may yield a reliable indication of the regression with much fewer observations than if the cases were selected at random. This principle is frequently applied in experimental or laboratory work, but is equally applicable in other types of investigations.

If the regressions are curvilinear, however, special selection of either extreme or central items of the independent variables forestalls the determination of the nature of the function, since curvilinear regressions can be determined only for the ranges of the independent factor, within which observations have been secured. For such regressions, therefore, the nature of the function may be more accurately determined if the independent items are so selected as to be spread fairly uniformly through the whole range of values, thus affording a sufficient number of observations for accurate determination of the nature of the relation throughout the whole range. Selection purely at random frequently provides more observations than are needed for certain portions of the curve, and provides so thin a scattering of observations at other portions as to make its true position and shape quite indeterminate, as has been illustrated previously. Even if curvilinearity is only suspected, such a uniform distribution of values for the independent variable provides an improved basis for determining whether or not the regression is truly linear, as compared with an equal number of observations selected at random. Where the dependent factor is normally distributed, selection with reference to the independent factor does not tend to distort the standard error of estimate, and for curvilinear functions, may give a truer value for it.

If the primary interest, however, is not in the nature of the relations and in determining how closely values of the dependent factor may be estimated (regressions and standard error); but instead is in determining what proportion of the original variation in the dependent factor can be accounted for on the basis of the relations determined (correlation and determination), then anything other than random selection with reference to any factor will destroy the value of the results. For most accurate results in such problems, the distribution of the dependent factor in the sample should be an accurate representation of the distribution in the universe from which the observations were drawn, and the only selection which would be justified would be aimed at securing such a sample.

Since the correlation coefficient or index, and the parallel measures of determination, are of significance only with respect to the standard deviation of the observed values of the dependent factor, it follows that when the dependent factor has such an abnormal distribution that its standard deviation is of little value as a descriptive statistic, the measures of correlation also tend to be of little value. For that reason correlation coefficients and indexes should be computed and used only when the dependent variable does have something approaching a normal distribution. For any series which actually yielded such an extreme distribution as the dichotomous values used in the third case of those just illustrated, measures of correlation would have little significance except their formal mathematical definition. Yet the regressions and standard errors of estimate would tend to retain all of their usual value and significance, so long as no selection had been made with reference to values of the dependent variable. In such a case, attempting to select the values of the dependent factor so as to make the series more nearly normal might seriously bias the regression results.

Accuracy of Observations.—The data with which the statistician has to deal are frequently subject to errors of observation. If corn yields are being studied in relation to fertilizer applications, for example, farmers may be able to estimate the yield per acre on a given tract only to within 5 or 10 bushels of the true yield. If livestock prices are being studied, the market reporter may not be able to get his daily average nearer than within 10 or 25 cents per 100 pounds of the true average of all the sales for the day. Or if educational ratings are being studied, the instructor may not be able to grade the test papers nearer than to within 5 or 10 per cent of the grade each really deserves. All of these illustrations are akin to the difficulties of the surveyor, who finds he cannot measure his angles more accurately than within a certain number of seconds; or of the astronomer, who finds his repeated observations disagree from each other by fractions of a second. But the errors of measurement are ordinarily tremendously greater in biological, economic, or social investigations than in physical observations; and for that reason statisticians must be particularly careful to use their data in such a way as to minimize the influence, upon their conclusions, of the errors which may be present.

Errors of observation may be such that they are not correlated with the value being observed, and hence tend to fall equally above and below the true values throughout the range of the variable; or else they may be such that they are correlated with the variable, tending usually to make the observed value fall above the true value in the upper part of the range, and below the true value in the lower part; or *vice versa.*

In correlation problems, there are two sets of true values involved, those for the dependent and independent variables; and there may also be two sets of errors, one tending to cause the observed values for the dependent variable to differ from the true values, and the other affecting the independent variable. The extent to which such errors, if present, modify or impair the results of correlation analysis depends both upon the type of the errors and the variables which they affect.

If the errors affect only values of the dependent factor, and if they are not correlated with the true values, their presence tends to lower the correlation and to increase the standard error of estimate, but does not tend to change the slope of the regression line from the true slope for the universe. If, however, uncorrelated errors are in the independent factor, that not only tends to lower the correlation and increase the standard error of estimate, but also tends to decrease the regression below the true value. Both of these cases may be illustrated from the same set of data used before.

Errors in the dependent variable.—The data used in Table 76 may be modified by assuming some random error influences Y, making one-third of the values 1 unit higher, one-third 1 unit lower, and leaving one-third unchanged. With these changes, the data appear as follows:

TABLE 77

Correlation Table, Showing Hypothetical Frequencies at Specified Values, With Random Errors in Y.

Values of Y	Values of X				
	0	1	2	3	4
−1	....	....	1	....	1
0	....	1	....	2	....
1	1	....	3	....	1
2	....	2	2	3	....
3	1	2	3	1	1
4	....	....	1	....	....
5	1	1	....	....	....

For these data, $\bar{r}_{yx} = -0.33$, $b_{yx} = -0.50$, and $\bar{S}_{yx} = 1.46$. The introduction of the random error into Y has lowered the correlation from that of -0.47 for the original values and increased the standard error of estimate; but it has had no significant effect upon the regression of Y on X, the new value --0.50 being identical with the value of -0.50 for the original data in Table 76.

Error in the independent variable.—If, however, X is regarded as the dependent factor, and Y as the independent, the regression coefficient for the new values, $b_{xy} = -0.28$, is found to be much reduced from that of -0.50 for the original values. Introducing even random errors into the observations of the independent factor markedly reduces the observed regressions below the true value.

The errors considered to this point have all been random errors. If instead the errors are correlated with either of the factors, then their presence would obscure the true relationship, and bias any correlation constants which might be computed, tending to make them either too high or too low, depending on the inter-relations between the errors and the variables.

Errors in both variables.—If random errors are associated with both variables simultaneously, then their effects are a blending of those just illustrated, tending to reduce both the closeness of correlation and the regression below the true values. For example, if random errors of the same magnitude are introduced into X as well as Y of Table 76, the values appear as follows:

TABLE 78

Correlation Table, Showing Hypothetical Frequencies at Specified Values, With Random Errors in Both X and Y

Values of Y	Values of X						
	−1	0	1	2	3	4	5
−1	....	....	....	....	2		
0	....	....	1	1	1		
1	1	....	1	1	1	1	
2	....	1	2	2	1	1	
3	....	1	2	2	1	1	1
4	....	....	....	....	1		
5	....	1	1				

With these changes, the correlation is reduced to practically 0, the standard error increased to 1.524, and the regression of Y on X changed to — 0.179. The comparison of these constants with those for the original data in Table 76 illustrates the extent to which the presence of random errors in the observed values of the variables may reduce the accuracy and effectiveness of correlation analysis.

Errors of Observation in Multiple Correlations.—The points which have been illustrated here for simple correlation are equally true for multiple correlation, both with respect to the influence of selection of sample and of the effect of errors of observation. The influence of errors of observation in multiple correlation problems may be illustrated by a case based on actual economic data.

Over the 17 years from 1907 through 1923, the monthly price of lambs shows a very high correlation with the price of wool and the price of dressed lamb. Using X_2 for the price of wool, in cents per pound, X_3 for prices of dressed lamb, in cents per pound, and X_1 for prices of live lambs, in cents per pound, multiple correlation gives for the 204 observations, $R_{1.23} = 0.991$, and $x_1 = 0.144x_2 + 0.354x_3$.

To test what effect random errors would have had on this correlation, two dice were thrown 204 times, giving random values from 2 to 12. These values were then added to the successive values of the dependent, and a similar set of 204 values to the successive observations of one independent factor, to see what effect it would have on the results. In the following tabulation the notation " $X + e$ " is used to designate the variables to whose values these " random errors " had been added.

EFFECT OF INTRODUCING RANDOM ERRORS ON CORRELATION RESULTS

Independent variables	Dependent variable	Multiple correlation	Regression equation
X_2 and X_3	X_1	0.991	$0.144x_2 + 0.354x_3$
X_2 and X_3	$X_1 + e$	0.821	$0.112x_2 + 0.424x_3$
X_2 and $X_3 + e$	X_1	0.953	$0.163x_2 + 0.277x_3$
X_2 and $X_3 + e$	$X_1 + e$	0.804	$0.152x_2 + 0.306x_3$

These results illustrate the principles just set forth. The introduction of random errors into the dependent variable (X_1) reduces the correlation, but does not greatly change the size of the two regression coefficients. It would appear, especially from the amount of the reduction in the net regression on X_2, that the errors in this case may not have been completely randomly distributed and uncorrelated with X_1, even though determined by throws of dice.

But the second modification, where the error is introduced into the independent variable X_3 instead, is much more striking. The correlation is not reduced so much as in the first case, and the regression of X_1 on X_2 is changed only slightly from the original value—and increased as it happens. The net regression of X_1 on $X_3 + e$, however, is only $\frac{3}{4}$ as large as was the net regression of X_1 on X_3, in spite of the fact that the error introduced was only enough to raise the standard deviation of X_3 from 6.14 to 6.64.

The final case, with errors introduced into both X_1 and X_3, shows the lowest correlation of any, as would be expected. The net regression of $X_1 + e$ on X_2 is but little different from what the regression of X_1 on X_2 was, whereas the net regression of $X_1 + e$ on $X_3 + e$, though larger than what the regression of X_1 on $X_3 + e$ was, is still definitely lower than the regression of X_1 on X_3. The regression equation in this last case, where $X_1 + e$ is the dependent, is not greatly different from what it was in the preceding case with X_1 as the dependent, in spite of the fact that one of the independent variables—X_3—had a significant random error of observation in its values both times.

These cases illustrate the extent to which random errors may confuse the true relations, if they are allowed to creep into the observations. Just how great an effect upon the results such random errors will have depends upon the magnitude of the errors, the original variations in the variables, and the closeness of the inter-correlation. Although equations can be derived to show how great a reduction in correlation errors of a given magnitude will produce, they are of little practical use in economic work, since it is usually difficult enough to determine whether there are errors of observation or not, much less to determine what magnitude they have.[1]

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[1] In the problem given, the significant values determining the effect of the errors are as follows:

$\sigma_1 = 3.96$ $\qquad\sigma_{1+e} = 4.74$

$\sigma_3 = 6.14$ $\qquad\sigma_{3+e} = 6.64$

For errors in the dependent variable, the relations between the true and the apparent correlation are indicated by the equation:

$$R^2_{(1+e).23\ldots n} = \frac{(R^2_{1.23\ldots n})\sigma_1^2}{\sigma_1^2 + \sigma_e^2} = R^2_{1.23\ldots n}\frac{1}{1 + \frac{\sigma_e^2}{\sigma_1^2}}$$

This gives what the new correlation would be if the errors were truly random, so that the new regression equation came out as identical with the old. In the problem given, this gives an expected value for R of 0.827, as compared to the 0.821 actually obtained.

The practical significance of the principles which are stated here is that if there is known to be a large but random error in observing some variable, that variable may still be used as the dependent variable in a correlation study without making the regressions or estimating equation very far wrong, if determined with a large number of cases; but on the other hand, any use of that variable as an independent variable will be certain to yield results which under-state the actual relations.

In cases where the errors are biased, they tend to make the results of correlation analysis more or less in error, quite regardless of the variables to which they apply. If the errors tend either to magnify or to minimize the differences which actually exist, they will have a parallel effect on the regression coefficients if they apply to the dependent variables, and an inverse effect if they apply to an independent variable. There are so many different types of bias, however, that no more definite statement of the effects can be laid down.

Random errors have the same type of effect in the case of curvilinear correlation that they do in linear correlation, since if they are truly random they will tend to be balanced out along all the portions of the regression curve alike, if in the dependent variable; or tend to confuse the relations along the curve, if in the independent variable; and so reduce the differences observed.

Biased errors, on the contrary, may happen to be concentrated along certain portions of the range, and hence have a much more marked effect at one point than at another. Although this might seriously disturb the significance of the curve, it probably would have an equally disastrous effect on the reliability of the straight line. About the only real difference between linearity and curvilinearity with regard to errors is that random errors in the dependent variable could be ''balanced out'' in the case of a straight line regression with a somewhat smaller number of observations than would be necessary to secure valid results for a curvilinear regression.

Where, with random errors in the dependent factor, there are not enough cases available to ''balance them out,'' the effect of the errors is to throw a varying amount of error into the conclusions, the exact amount of the error depending on how closely the errors approach being canceled out. The illustrative case, where with over 200 observations the regressions were still changed somewhat, probably indicates what may be obtained by a combination of slight departures from true ''randomness'' in the errors with a sample not quite large enough to eliminate, entirely, all of the resulting instability. This may be nearer to what would usually happen in practice than the theoretical complete elimination of the errors in the dependent variable.

Summary.—Modification of the observations from the true conditions, either by selection of the sample or by the presence of errors of observation, tends to alter the value of the coefficient of correlation. If the regression line or curve is of primary interest, however, its accuracy of determination may be increased by suitable selection of observations with respect to independent factors. Similarly, random errors of observation may not influence the regressions, if the factor they affect can be treated as the dependent factor, and if enough observations are available to balance out the errors. These points hold true for multiple correlation problems as well as for 2-variable problems.

CHAPTER 20

MEASURING THE RELATION BETWEEN ONE VARIABLE AND TWO OR MORE OTHERS OPERATING JOINTLY

In working out the change in one variable with changes in other variables up to this point we have assumed that the relation of the dependent factor to each independent factor did not change, no matter what combination of other independent factors was present. In the case of the yield of corn, for example, as worked out in Chapter 14, we assumed that the effect of a given change in rainfall upon the yield was the same, no matter what was the temperature for the season. The significance of this assumption may be shown by combining the estimate for rainfall with the estimate for temperature, and plotting the combined influence of the two variables. In Table 68 we already have this combined influence worked out, so all we have to do is to plot it. Figure 53 shows the resulting figure. In reading this figure it should be noted that the inches of rainfall are read along the right-hand edge of the bottom of the cube, the degrees of temperature along the left-hand edge, and the yield along the vertical edge. The yield for any combination of temperature and rainfall is then shown by the distance the upper surface of the solid figure is above the point of intersection of the corresponding values in the base plane.[1]

Inspecting Figure 53, we can now see what is meant by saying that the changes in yield are assumed to be the same for each change in rainfall, no matter what the temperature. As shown in the figure, the maximum yield with a temperature of 70° is obtained at about 12 inches of rain—and that is also the rainfall which produces a maximum yield with a temperature of 72°, 74°, or 78°. Each curve has just exactly the same shape, and the only difference is their elevation above the base. On looking at it the other way, the same is true of temperature—for 9 inches rainfall the maximum yield is obtained at about 75° temperature—and the same is true for the other rainfall curves. This relation necessarily follows the assumptions made in measuring it. Figure 53 merely shows the estimate we get by the use of equation (54)

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[1] The way this figure is made may be thought of as follows: Suppose we drew a series of charts of the estimated differences in yield with differences in rainfall, with one chart for an average temperature of 70°, one for 72°, one for 74°, etc. Then if we cut these charts off at the yield line, and arrange them one back of the other, at even distances, we would have a figure looking much like Figure 52. The lines sloping across the surface from left to right represent what would be the tops of this series of charts.

$$X_1 = a + f(X_2) + f(X_3)$$

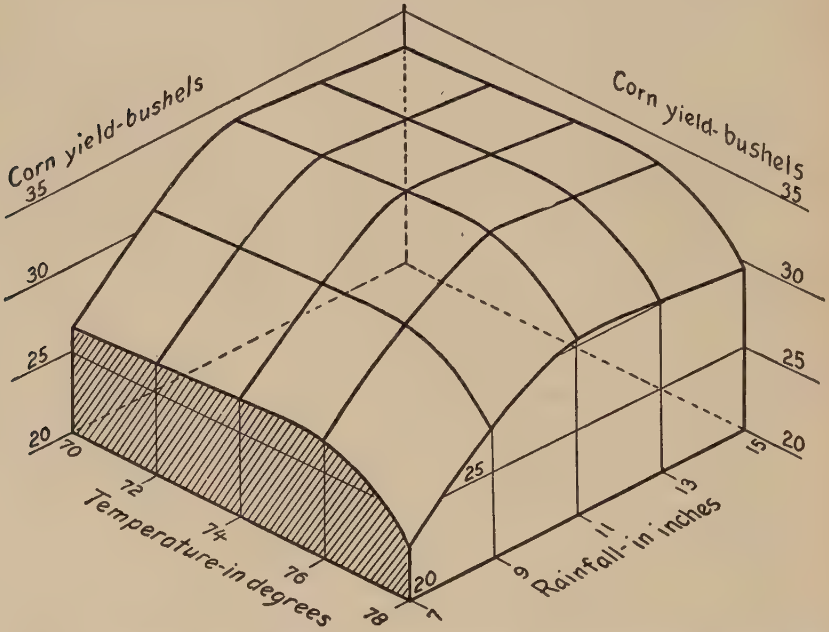

FIG. 53.—Probable yield of corn for various specific combinations of rainfall and temperature, from multiple curvilinear correlation.

In working out these estimates we simply add together the estimated value for X_2 and the estimated value for X_3. It does not make any difference what the value of X_3 is, the changes in X_1 assumed to accompany particular changes in X_2 are the same—and that is what the figure shows.

Only a little reflection is needed to indicate that Figure 53 may not tell the whole truth of the relation of yield to rainfall and temperature. It is quite possible that the crop can use more rain in a hot season than in a cool one, so that the rainfall which will produce the maximum crop may be higher in a season of high average temperature than in a season of low temperature. If that is really the case, equation (54) is unable to express the relationship, for as just pointed out, that equation assumes that the change in yield with rainfall is the same, no matter what the temperature.

An extreme illustration of a changing relationship is shown in Figure 54. This figure, which is based on actuarial investigations,[2] shows the differences in mortality among men from the usual rate, for differences in weight at different ages. Taking the 22-year line, for example, the figure shows that men who are much over normal weight have a much higher mortality than normal for that age. Then as the weight is less the mortality is less, until at normal weight there is only normal mortality. But as the weight drops still more, the mortality increases again, until below 80 per cent of the normal weight, the mortality is more than 20 per cent in excess of normal.

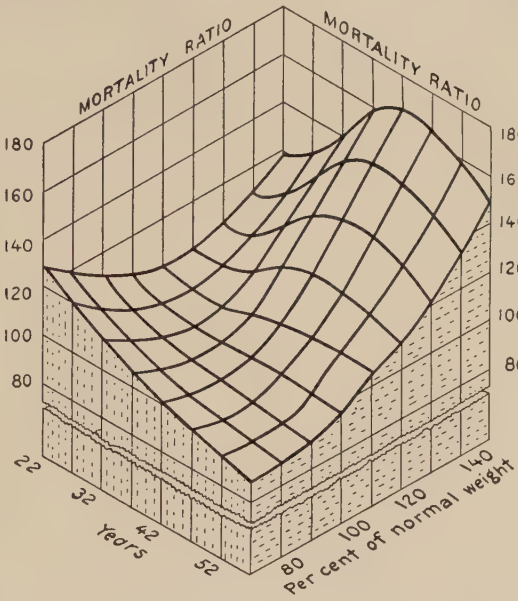

FIG. 54.—Differences in mortality with differences in weight, for men of various ages. (Each in per cent of average mortality for that age.) Illustration taken from an article by Andrew Court.

The relation is somewhat different for 52-year-old men, however. For them the mortality is also higher for those who are above normal in weight, and decreases as normal weight is reached. But as the weight falls below normal the mortality continues to decrease, until for men who are only 70 per cent of normal weight, the mortality is more than 15 per cent *below* the normal for that age. For ages intermediate between these two, the change is also intermediate—as is shown in the chart, 27 years is similar to 22, but not so marked, while the line for 47 years is similar to that for 52. At 42 years, there is apparently little difference in mortality anywhere between 70 per cent of normal weight and 100 per cent.

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[2] Medico-Actuarial Investigations, 1913, Vol. II, p. 24.

Figure 54 illustrates a situation which the previous methods of analysis would be quite incapable of dealing with adequately. Were equation (54) used to represent this relation, the higher mortality with lower weights for young men would tend to balance out the lower mortality with lower weights with the older men. In fact, the erroneous conclusion might be reached that the age does not affect the mortality at all. Figure 55 shows the results of an attempt to represent this relation by the methods previously discussed. It is quite obvious that the results fall far short of the relations as shown by Figure 54.

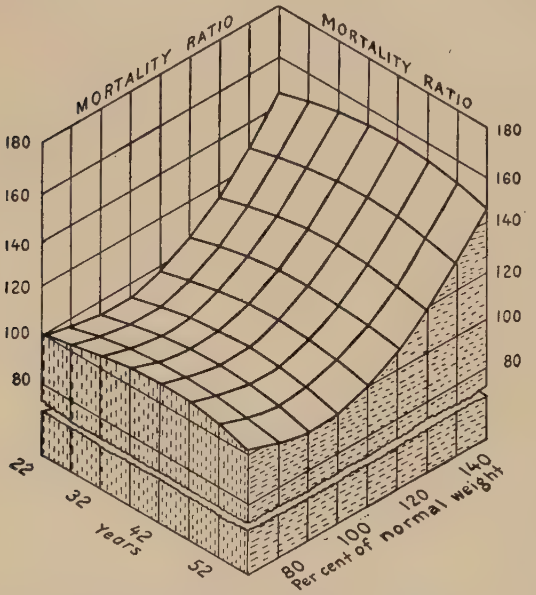

FIG. 55.—Relation shown in previous figure, represented by equation $X_1 = f(X_2) + f(X_3)$.

Use of "Joint Functions" to Show Combined Effects.—What is needed in both the corn yield problem and the mortality problem is some way of determining what the yield, in the one case, or the mortality, in the other, is most likely to be for any given *combination* of the two independent variables. That is quite different from asking for the separate effect of each one. Obviously, a small change in one independent factor will be expected to be accompanied by only a small change in the dependent, so that all the estimated yields (or mortalities) will be expected to lie along a continuous surface like that shown in Figure 53 or 54; but the surface will be free to warp or change its shape in different portions like the surface shown in Figure 54, instead of being held rigidly to the same shape in each dimension, like the surfaces in Figure 53 or 55. Mathematically, such a changing relation between one variable and two or more others is known as a *joint functional relation,* and may be indicated by the equation:

$$X_1 = f(X_2, X_3) \tag{75}$$

This is read simply that "X_1 is a joint function of X_2 and X_3." That means only that for any combination of values of X_2 and X_3, there will be some particular value of X_1. Equation (75) is therefore capable of representing either a relation such as that shown in Figure 53, or the more complex relation shown in Figure 54.

The problem of determining the extent to which corn yield varies with the joint effect of temperature and rainfall may be said to be one of determining the functional relation of yield to the two other factors, according to the relation shown in equation (75).

Determining a Joint Function for Two Independent Variables.—Where only two independent variables are concerned, the joint functional relation may be determined quite simply, if a large enough number of observations are available.

The process may be illustrated by data from a different problem, shown in Table 79. The observations are from a field study of haystack dimensions in the Great Plains area. Farmers in this area ordinarily sell their hay unbaled and in the stack. It is therefore necessary to estimate the quantity of hay each stack contains. Two measurements, which can be made readily with only a rope, are usually employed—the perimeter around the base of the stack, and the "over," or the distance from the ground on one side of the stack over the center, to the ground on the other. The observations shown in Table 79 are all for round stacks. These stacks vary in height and shape to some extent, however, so their volume cannot be computed from the basal circumference by any simple mathematical rule, such as for the volume of a hemisphere. The volumes shown in the table are computed from careful surveying measurements of all the dimensions of each stack—much more exact measurements than a farmer would be able to make in practice. The problem is to establish the average volume for specified circumferences and "overs," so the farmers may be able to use these two measurements, and also to determine how much confidence can be placed in estimates of volume based on these two factors.

The volume will tend to be some function of the basal area times the height. The basal area is a function of the square of the basal circumference; the "over" is a function of both the basal diameter and

TABLE 79

DATA TAKEN FROM NEBRASKA ROUND STACKS MEASURED IN 1927 AND 1928

Volume, in cubic feet	Circumference, in feet	"Over," in feet	X_2*	X_3*	X_1*	X'_1	z
2853.00	69.0	37.00	0.139	0.168	0.455	0.478	−0.023
2702.00	65.0	36.50	0.113	0.162	0.432	0.450	−0.018
3099.00	73.0	38.50	0.163	0.185	0.491	0.447	0.044
1306.00	62.5	26.50	0.096	0.023	0.116	0.143	−0.027
2294.00	70.0	35.00	0.145	0.144	0.361	0.436	−0.075
2725.00	68.0	36.50	0.133	0.162	0.435	0.421	0.014
3309.00	71.0	39.25	0.151	0.194	0.520	0.557	−0.037
2790.00	64.0	36.75	0.106	0.165	0.446	0.450	−0.004
2756.00	62.0	38.50	0.092	0.185	0.440	0.478	−0.038
5237.92	80.0	43.00	0.203	0.233	0.719	0.705	0.014
3149.82	67.0	37.60	0.126	0.175	0.498	0.490	0.008
5498.46	79.0	44.60	0.198	0.249	0.740	0.739	0.001
3397.83	66.0	38.00	0.120	0.180	0.531	0.541	−0.010
3007.56	62.0	36.80	0.092	0.166	0.478	0.486	−0.008
4574.29	79.0	41.10	0.198	0.214	0.660	0.596	0.064
6228.59	73.0	48.00	0.163	0.281	0.794	0.780	0.014
2318.64	63.0	30.20	0.099	0.080	0.365	0.265	0.100
3176.71	68.0	37.75	0.133	0.177	0.502	0.502	0
2352.31	70.0	32.50	0.145	0.112	0.371	0.363	0.008
2174.44	69.0	31.62	0.139	0.100	0.337	0.333	0.004
2694.72	73.0	34.50	0.163	0.138	0.431	0.433	−0.002
3333.53	70.0	37.25	0.145	0.171	0.523	0.500	0.023
4328.92	78.5	40.00	0.195	0.202	0.636	0.617	0.019
2115.04	67.0	31.25	0.126	0.095	0.325	0.317	0.008
2489.08	66.5	33.75	0.123	0.128	0.396	0.388	0.008
2296.65	64.5	32.38	0.110	0.110	0.361	0.338	0.023
3117.21	65.5	37.58	0.116	0.175	0.494	0.480	0.014
4088.36	74.0	40.33	0.169	0.206	0.612	0.602	0.010
4180.88	72.0	40.50	0.157	0.207	0.621	0.594	0.027
2318.19	63.0	33.00	0.099	0.119	0.365	0.346	0.019
1946.90	58.0	31.00	0.063	0.091	0.289	0.255	0.034
2479.89	61.0	36.50	0.086	0.162	0.394	0.423	−0.029
3174.80	73.0	37.00	0.163	0.168	0.502	0.506	−0.004
2151.54	64.0	33.00	0.106	0.119	0.333	0.353	−0.020
3475.68	73.0	39.50	0.163	0.197	0.541	0.576	−0.035
4393.08	71.0	42.00	0.151	0.223	0.643	0.624	0.019
2819.50	69.0	35.00	0.139	0.144	0.450	0.432	0.018
3703.49	70.0	38.50	0.145	0.185	0.569	0.530	0.039
2742.81	72.5	34.50	0.160	0.138	0.438	0.430	0.008

* $X_2 = \log_{10}$ (circumference) − 1.700, stated to three decimal places.
$X_3 = \log_{10}$ ("over") − 1.4, stated to three decimal places.
$X_1 = \log_{10}$ (volume) − 3.0, stated to three decimal places.

† Acknowledgment is due W. H. Hosterman, of the Bureau of Agricultural Economics, U. S. Dept. of Agriculture, for the use of these data.

TABLE 79—*Continued*

Volume, in cubic feet	Circumference, in feet	"Over," in feet	X_2*	X_3*	X_1*	X'_1	z
3002.40	66.0	35.50	0.120	0.150	0.477	0.430	0.047
1854.19	69.0	30.50	0.139	0.084	0.268	0.297	−0.029
1982.07	62.0	31.00	0.092	0.091	0.297	0.288	0.009
2470.86	65.0	33.50	0.113	0.125	0.393	0.373	0.020
1203.15	60.1	26.25	0.079	0.019	0.080	0.117	−0.037
2843.84	71.0	36.00	0.151	0.156	0.454	0.469	−0.015
2636.25	66.0	36.00	0.120	0.156	0.421	0.443	−0.022
1998.39	65.0	32.00	0.113	0.105	0.301	0.330	−0.029
2005.03	64.0	32.00	0.106	0.105	0.302	0.323	−0.021
2568.76	66.0	35.00	0.120	0.144	0.410	0.418	−0.008
2161.18	65.0	32.50	0.113	0.112	0.335	0.345	−0.010
2112.20	67.0	32.00	0.126	0.105	0.325	0.333	−0.008
3009.33	65.0	38.00	0.113	0.180	0.478	0.438	0.040
1992.24	63.0	31.00	0.099	0.091	0.299	0.288	0.011
2746.98	70.0	34.00	0.145	0.131	0.439	0.407	0.032
2238.27	64.0	35.00	0.106	0.144	0.350	0.406	−0.056
1747.47	67.0	30.00	0.126	0.077	0.242	0.280	−0.038
2863.91	67.0	36.00	0.126	0.156	0.457	0.448	0.009
3593.47	72.0	39.00	0.157	0.191	0.555	0.555	0
2435.48	62.0	35.00	0.092	0.144	0.387	0.443	−0.056
2430.18	63.0	34.00	0.099	0.131	0.386	0.362	0.024
2590.07	67.0	35.00	0.126	0.144	0.413	0.423	−0.010
3577.68	70.0	41.00	0.145	0.213	0.554	0.596	−0.042
3299.24	73.0	40.00	0.163	0.202	0.518	0.598	−0.080
1986.14	64.0	32.50	0.106	0.112	0.298	0.338	−0.040
3109.04	68.0	38.00	0.133	0.180	0.493	0.508	−0.015
2821.56	71.0	37.00	0.151	0.168	0.450	0.498	−0.048
2932.24	67.0	38.00	0.126	0.180	0.467	0.501	−0.034
3304.63	69.0	38.00	0.139	0.180	0.519	0.514	0.005
2565.46	72.0	35.00	0.157	0.144	0.409	0.450	−0.041
4509.93	74.0	41.33	0.169	0.216	0.654	0.627	0.027
4804.01	81.0	42.00	0.208	0.223	0.682	0.683	−0.001
4241.80	75.0	40.75	0.175	0.210	0.627	0.619	0.008
4516.10	69.2	43.25	0.140	0.236	0.655	0.643	0.012
5011.62	77.5	43.10	0.189	0.234	0.700	0.691	0.009
2110.73	65.0	31.50	0.113	0.098	0.324	0.316	0.008
2775.70	76.0	34.60	0.181	0.139	0.443	0.448	−0.005
3927.90	72.0	39.00	0.157	0.191	0.594	0.555	0.039
4212.77	80.0	41.50	0.203	0.218	0.624	0.663	−0.039
3562.64	78.5	38.50	0.195	0.185	0.552	0.575	−0.023

* $X_2 = \log_{10}$ (circumference) − 1.700, stated to three decimal places.
$X_3 = \log_{10}$ ("over") − 1.4, stated to three decimal places.
$X_1 = \log_{10}$ (volume) − 3.0, stated to three decimal places.

TABLE 79—*Continued*

Volume, in cubic feet	Circumference, in feet	"Over," in feet	X_2*	X_3*	X_1*	X'_1	z
2853.96	75.0	35.50	0.175	0.150	0.455	0.461	−0.006
3294.38	69.0	38.00	0.139	0.180	0.518	0.514	0.004
1689.54	63.0	30.50	0.099	0.084	0.228	0.274	−0.046
2228.84	62.0	33.00	0.092	0.119	0.348	0.341	0.007
2362.61	64.0	34.00	0.106	0.131	0.373	0.379	−0.006
3088.28	68.0	38.50	0.133	0.185	0.490	0.520	−0.030
3820.79	70.0	40.00	0.145	0.202	0.582	0.570	0.012
3126.64	63.0	36.90	0.099	0.167	0.495	0.447	0.048
3624.75	71.0	38.45	0.151	0.185	0.559	0.536	0.023
3023.97	73.0	36.50	0.163	0.162	0.480	0.493	−0.013
6045.42	79.0	47.00	0.198	0.272	0.781	0.798	−0.017
3100.11	64.0	37.00	0.106	0.168	0.491	0.457	0.034
3378.07	70.0	38.00	0.145	0.180	0.529	0.519	0.010
3040.29	77.0	35.00	0.186	0.144	0.483	0.464	0.019
2252.16	65.0	32.50	0.113	0.112	0.353	0.345	0.008
3552.61	76.0	37.00	0.181	0.168	0.551	0.481	0.070
2635.90	66.0	34.50	0.120	0.138	0.421	0.405	0.016
3201.41	71.0	35.50	0.151	0.150	0.505	0.455	0.050
2590.21	69.0	35.00	0.139	0.144	0.413	0.432	−0.019
3743.55	76.0	38.25	0.181	0.183	0.573	0.558	0.015
3858.03	73.0	39.50	0.163	0.197	0.586	0.576	0.010
3829.44	74.0	39.75	0.169	0.199	0.583	0.586	−0.003
2556.44	66.0	33.00	0.120	0.119	0.408	0.365	0.043
3119.07	69.0	36.00	0.139	0.156	0.494	0.460	0.034
2122.38	65.5	32.00	0.116	0.105	0.327	0.332	−0.005
2921.92	69.0	36.00	0.139	0.156	0.466	0.460	0.006
2936.35	72.5	34.50	0.160	0.138	0.468	0.430	0.038
2427.66	76.0	33.00	0.181	0.119	0.385	0.399	−0.014
2069.38	65.0	31.50	0.113	0.098	0.316	0.315	0.001
1899.54	72.0	30.00	0.157	0.077	0.279	0.285	−0.006
4289.28	78.5	40.50	0.195	0.207	0.632	0.629	0.003
2407.39	67.5	32.50	0.129	0.112	0.381	0.358	0.023
3097.99	66.0	35.50	0.120	0.150	0.491	0.430	0.061
3893.67	75.5	39.25	0.178	0.194	0.590	0.582	0.008
2238.66	68.0	31.75	0.133	0.102	0.350	0.336	0.014
2314.79	64.0	33.10	0.106	0.120	0.364	0.356	0.008
2667.07	66.0	34.70	0.120	0.140	0.426	0.409	0.017
2582.07	68.0	33.50	0.133	0.125	0.412	0.388	0.024
3426.50	75.0	37.00	0.175	0.168	0.535	0.516	0.019
2307.34	60.0	33.40	0.078	0.124	0.363	0.336	0.027
3960.41	76.0	39.30	0.181	0.194	0.598	0.585	0.013

* $X_2 = \log_{10}$ (circumference) − 1.700, stated to three decimal places.
$X_3 = \log_{10}$ ("over") − 1.4, stated to three decimal places.
$X_1 = \log_{10}$ (volume) − 3.0, stated to three decimal places.

the height—but attempts to separate the two have been unsuccessful. It is obvious, however, that any attempt to represent the relation by a regression equation of the type

$$\text{volume} = f\,(\text{circumference}) + f\,(\text{“ over ”})$$

will be unsatisfactory because of the multiplying nature of the relations, that is,

$$\text{volume} = f\,(\text{circumference})(\text{over})$$

Such a relationship may be approached by use of the relation

$$\log_{\text{volume}} = f\,(\log_{\text{circumference}}) + f\,(\log_{\text{over}})$$

Attempts to determine the relationship by this equation, however, have not been fully successful. The shape of the stacks apparently shifts with changes in size.

The haystack problem is evidently one where the relation may best be expressed by a joint function such as

$$\text{volume} = f\,(\text{circumference, over})$$

Such a relation could be determined directly from the data by the methods which will presently be described. It is evident that the correlation surface would have a marked upward slope as the two dimensions increased together, even if the usual volume formulas applied. The work for this particular problem may be somewhat simplified by first stating each variable as a logarithm and then determining the joint relation according to the equation

$$\log_{\text{volume}} = f\,(\log_{\text{circumference}}, \log_{\text{over}})$$

The logarithms (to base 10) are accordingly also shown in Table 79, and are designated as X_2, X_3, and X_1. (To facilitate the subsequent computations, 1.7 has been subtracted from the logarithm for circumference, 1.4 from the logarithm for “ over,” and 3.0 from the logarithm for volume.)

Subgrouping and averaging the observations.—The first step in the process of determining the joint functional relation is to classify the observations according to X_2, and subclassify according to X_3, and determine the averages of X_1, X_2, and X_3 for each group. Since there are only 120 observations, it would not be worth while to make too many groups. Four groups each way would give 16 subgroups, and 5 each way would give 25. If the cases were uniformly distributed through 25 subgroups, that would make less than 5 cases to a group, which is rather thin for a satisfactory average (though it might be sufficient in this particular problem, where the correlation is much higher than in many problems which must be dealt with.) However, the cases will not necessarily be distributed uniformly through all the groups, so it will be best if we try the 5-fold classification and see how the cases fall.

TABLE 80

Haystack Observations, Classified According to X_2 and X_3

(Logarithms of Circumference and "Over")

X_3 values	X_2 values				
	Under 0.090	0.090–0.119	0.120–0.149	0.150–0.179	0.180 and over
Under 0.100	2	7	3	1	
0.100–0.139	1	14	10	3	2
0.140–0.179	1	8	17	8	2
0.180–0.219	....	2	10	14	7
0.220 and over	....	....	1	2	5

There is a marked correlation between X_2 and X_3, so a few groups have 10 or more reports, while 15 out of the 25 have under 5. Preliminary examination of the data indicates that a unit change in X_3 is generally accompanied by a larger change in X_1 than is a unit change in X_2. Accordingly we may decide to halve the groups in the central portion of the range of X_3, making the class intervals with respect to that variable under 0.100, 0.100 – 0.119, 0.120 – 0.139, 0.140 – 0.159, 0.160 – 0.179, 0.180 – 0.199, 0.200 – 0.219, and 0.220 and over. With 5 classes for X_2, this will give a 40-group classification—but with many of the "cells" vacant. Averaging X_2, X_3, and X_1 for each of the resulting groups gives averages as shown in Table 81.

Plotting the Subgroup Averages, and Drawing First Approximation Curves.—Inspection of the averages of X_2 down each column in Table 81 shows that most of the variation in that factor has been eliminated, except in the upper subgroups of X_3, above $X_3 = 0.200$, where the averages tend to fall above the mean of the range. There is a more marked tendency for the averages of X_3 to rise across the rows from left to right. Accordingly the groups classified with respect to X_2 will be studied first, to determine the changes in X_1 with changes in X_3, X_2 being held (approximately) constant at various values. This may be done by plotting separately the average difference in X_1 with differences

TABLE 81

HAYSTACK DATA: AVERAGE X_2, X_3, AND X_1, FOR OBSERVATIONS CLASSIFIED BY X_2 AND X_3

X_3 values	Number of cases	X_2 under 0.090		
		Mean X_2	Mean X_3	Mean X_1
Under 0.100	2	0.071	0.055	0.185
0.100–0.119				
0.120–0.139	1	0.078	0.124	0.363
0.140–0.159				
0.160–0.179	1	0.086	0.162	0.394
		X_2 0.090–0.119		
Under 0.100	7	0.102	0.081	0.278
0.100–0.119	10	0.107	0.112	0.332
0.120–0.139	4	0.106	0.127	0.379
0.140–0.159	2	0.099	0.144	0.369
0.160–0.179	6	0.105	0.167	0.473
0.180–0.199	2	0.103	0.183	0.459
		X_2 0.120–0.149		
Under 0.100	3	0.130	0.085	0.278
0.100–0.119	6	0.132	0.108	0.362
0.120–0.139	4	0.130	0.131	0.417
0.140–0.159	12	0.129	0.149	0.440
0.160–0.179	5	0.135	0.171	0.483
0.180–0.199	8	0.135	0.181	0.515
0.200–0.219	2	0.145	0.208	0.568
0.220 and over	1	0.140	0.236	0.655
		X_2 0.150–0.179		
Under 0.100	1	0.157	0.077	0.279
0.100–0.119				
0.120–0.139	3	0.161	0.138	0.446
0.140–0.159	4	0.159	0.150	0.456
0.160–0.179	4	0.163	0.167	0.492
0.180–0.199	9	0.161	0.193	0.558
0.200–0.219	5	0.167	0.208	0.606
0.220 and over	2	0.157	0.252	0.719
		X_2 0.180 and over		
0.100–0.119	1	0.181	0.119	0.385
0.120–0.139	1	0.181	0.139	0.443
0.140–0.159	1	0.186	0.144	0.483
0.160–0.179	1	0.181	0.168	0.551
0.180–0.199	3	0.186	0.187	0.574
0.200–0.219	4	0.198	0.210	0.638
0.220 and over	5	0.199	0.242	0.724

in X_3, for each column. Figure 56 shows these averages, with the number of observations represented by each one indicated. Apparently the relation, for each group of averages, tends to be linear, so straight lines are drawn in by eye as first approximations to the final relation. It should be noticed, however, that these lines are not all of the same slope, but tend to slope more steeply as X_2 increases. In some problems curves instead of straight lines would be indicated by these group averages. In such a case, separate curves would be fitted free-hand to each set of averages. In drawing such curves it is desirable to keep them as nearly of the same shape as the data will permit, and to change the shape only gradually from one to the next.

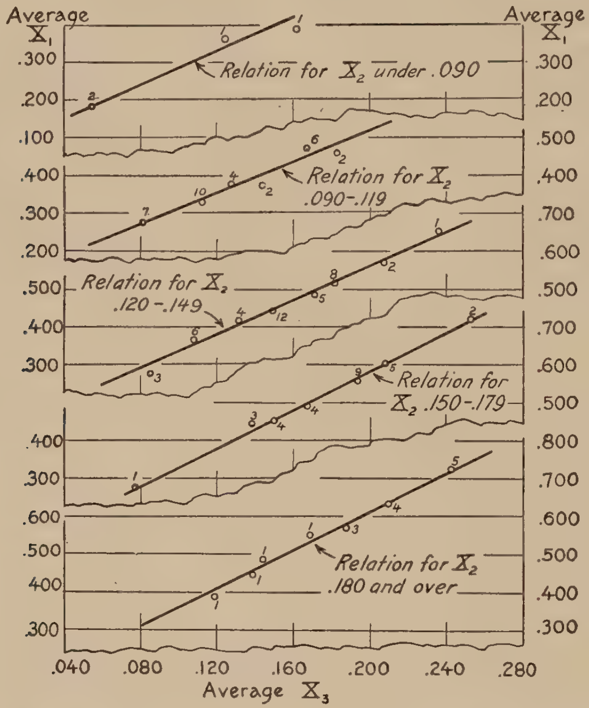

Fig. 56.—Differences in X_1 with differences in X_3 for specified values of X_2, and first approximate curves.

Obtaining a Second Approximation to the Joint Surface.—After the first approximation lines or curves have been drawn along the X_3 axis, the next step is to smooth along the X_2 axis. To do this the values of X_1 according to the first approximation curves are read off at intervals on X_3, corresponding to the central values of the groups of X_3 in Table 81. These values are shown in Table 82.

TABLE 82

ESTIMATED VALUES OF X_1 FOR SPECIFIED VALUES OF X_2 AND X_3, FROM FIRST APPROXIMATION LINES

X_3	X_2				
	Under 0.090	0.090–0.119	0.120–0.149	0.150–0.179	0.180 and over
0.060	0.195	0.233	0.247	....	....
0.110	0.307	0.334	0.359	0.354	0.382
0.130	0.352	0.374	0.403	0.404	0.433
0.150	0.397	0.415	0.448	0.454	0.484
0.170	0.442	0.455	0.492	0.505	0.535
0.190	....	0.496	0.537	0.555	0.586
0.210	....	....	0.583	0.605	0.637
0.240	....	....	0.650	0.681	0.713

The readings shown in Table 82 may next be smoothed along the X_2 axis by plotting the estimated value of X_1 for the specified values of X_3, with varying values of X_2. This process is shown in Figure 57. The values used as the coordinates for X_2 are the corresponding average values of X_2 for each subgroup in Table 81. Thus in plotting the values of X_1 for $X_3 = 0.060$—the first line in the table above—the values for X_2 are the averages from the first lines in Table 81—0.071, 0.102, 0.130, etc. By using these actual averages allowance is made for cases such as those noted previously, where not even the process of subgrouping has completely removed the influence of the other independent variable.

The averages plotted in Figure 57 show a slight but consistent curvilinear relation of X_1 to X_2, with a gradually increasing slope for the higher values. In two cases straight lines would fit as well as curves, but since all the remaining groups show consistent slight curves, they are drawn in here as well. The averages, smoothed free-hand along the X_2 axis, give a second approximation to the joint functional relation.

Making the final Smoothing of the Approximation Curve.—As a final check, values from the curves in Figure 57 may be read off for stated values of X_2, and smoothed again with reference to X_3. Since the variation in the averages of X_2 in each column of Table 81, and in the averages of X_3 in each row, have now been allowed for by the methods used in constructing Figures 56 and 57, the new readings may be taken for any convenient interval on X_2. The values 0.070, 0.080, 0.100 0.120, 0.140, 0.160, 0.180, and 0.200 may be used, giving convenient

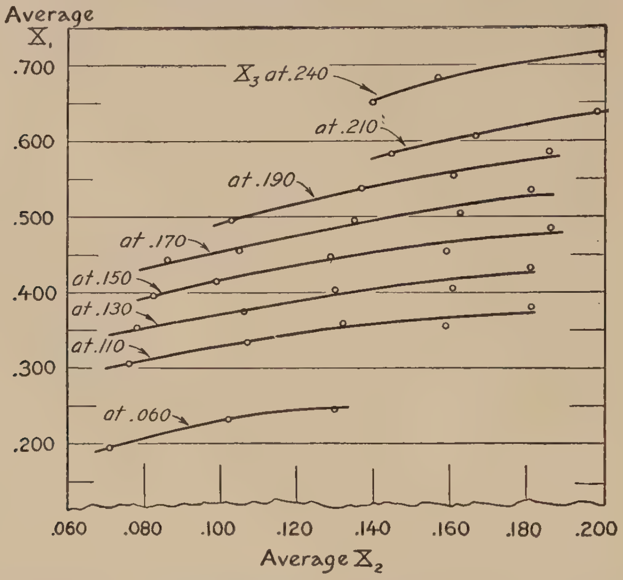

FIG. 57.—Differences in X_1 with differences in X_2 for specified values of X_3, read from smoothed curves in Fig. 56.

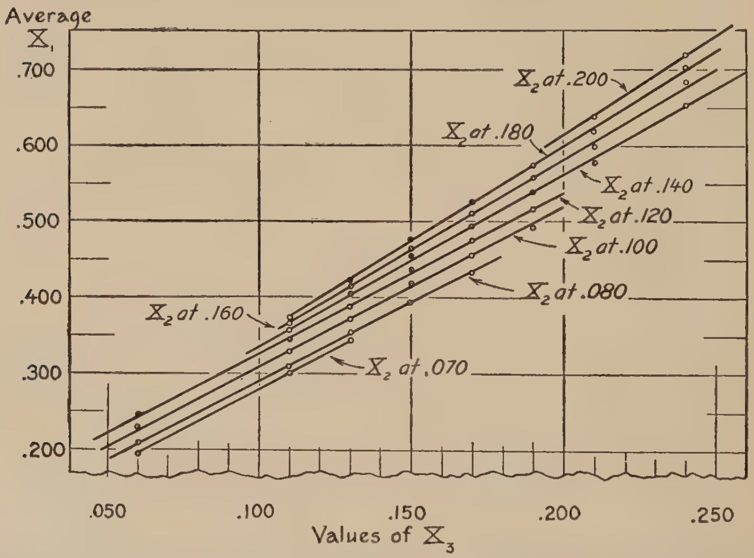

FIG. 58.—Differences in X_1 with differences in X_3, from second smoothed curves.

values for subsequent interpolations. Reading off the corresponding X_1 values from the curves in Figure 57, just shown before in Table 82, and plotting with the X_3 values as abscissas, gives the results shown in Figure 58.

After the X_1 readings from Figure 57 are plotted, as indicated by the hollow circles in Figure 58, they are smoothed with reference to the X_3 axis. It is found again that straight lines serve to describe the relation, and these are accordingly drawn in by eye, with some consideration of adjacent lines where otherwise the line would be out of agreement, as for $X_2 = 0.200$. These lines, showing the estimated values of X_1 for specified differences in X_2 and X_3, may be taken as defining the functional relation between the three variables. This figure indicates very clearly the "warping" of the regression surface. The increase in X_1 per unit increase in X_3 is much greater for large values of X_2 than for small values. That fact could not have been expressed in any regression equation of the form $X_1 = f(X_2) + f(X_3)$. The shape of the "correlation surface" may be seen in Figure 59, where the final lines from Figure 58 have been combined into a three-dimension diagram.

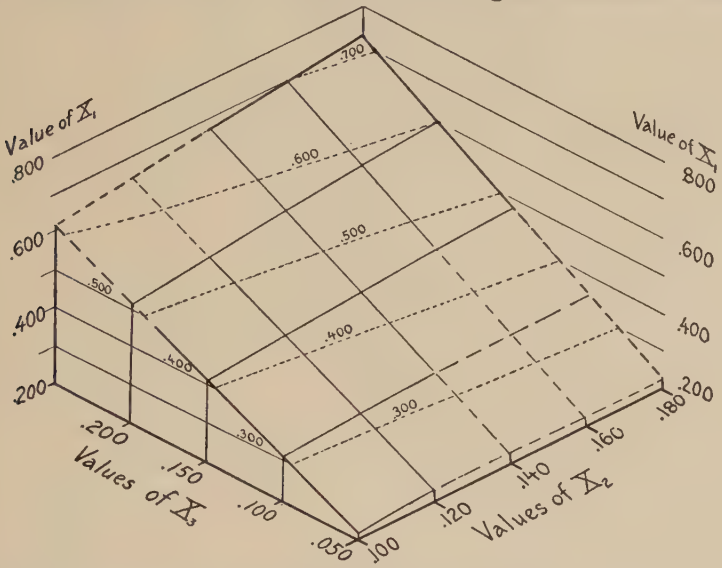

FIG. 59.—Probable value of X_1 for specific combinations of X_2 and X_3, from smoothed surface.

Estimating X_1 from the Joint Function.—Estimates of X_1 for any combination of values of X_2 and X_3 may be made directly from Figures

58 or 59 by making the necessary interpolations. The process may be more conveniently carried out by making a " contour chart," which shows the differences in X_1, for different combinations of X_2 and X_3, by a series of lines passing through combinations of the other two variables which will produce equal values of X_1. Thus if a series of planes were passed through the cube in Figure 59, parallel to the base plane, at $X_1 = 0.300, 0.400, 0.500$, etc., they would cut the top surface of the solid in the intersections indicated by the dotted lines. Then if one were to look straight down upon the top of the solid, these dotted lines would appear as shown in Figure 60. Other lines have been drawn in between these to indicate 0.050 differences in X_1.[3]

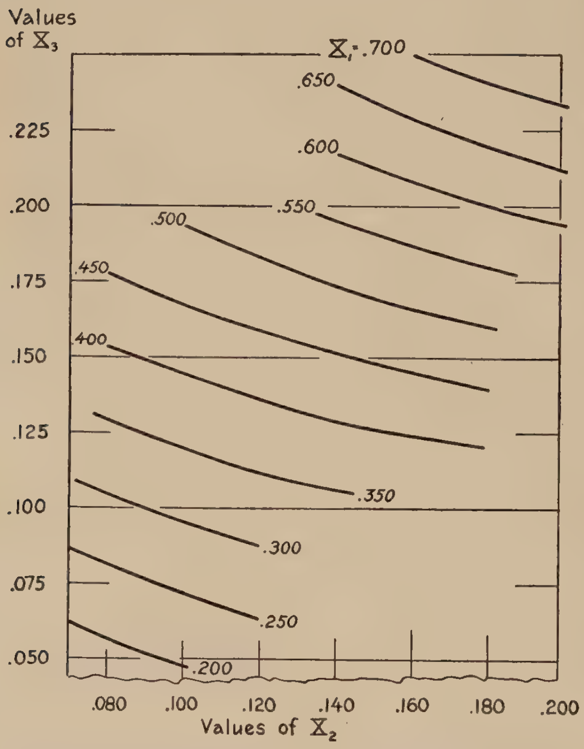

FIG. 60.—Probable value of X_1 for specific combinations of X_2 and X_3, shown by contours.

Determining the Standard Error of Estimate and Index of Multiple Correlation. — Estimated yields for each observation may now be worked out by the use of Figure 58 or 60, interpolating for the distances between the adjacent lines where an observation falls off the line. Table 79 also shows these estimated values (X'_1), and the difference between the actual and the estimated. The standard deviation of the original values of X_1 is 0.1265, while the standard deviation of the residuals computed in Table 79 is 0.0295. Apparently the regression surface accounts for almost all of the variation in volume. The accuracy with which estimates of X_1 may be made from X_2 and X_3 may be determined by adjusting σ_z in the usual way.

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[3] Figure 60 may be most readily drawn from Figure 58. By noting the value of X_3 necessary to produce a value of $X_1 = 0.300$, for each X_2 line, a series of values for X_2 and X_3 may be located, through which the contour for $X_1 = 0.300$ is drawn. The values for $X_1 = 0.400$ are then noted, giving the location of the 0.400 contour, and so on.

For the type of surface shown, the relations might be quite closely represented by an equation of the type

$$X_1 = a + b_2X_2 + (b_3 + b'_3X_2)X_3$$ [4]

This equation expresses the relation shown in Figure 59: for a constant value of X_2, the regression of X_1 on X_3 is linear; but as X_2 changes, this regression also changes at a uniform rate. The equation given has 4 constants, so in adjusting σ_z to determine the standard error of estimate, $m = 4$. Hence

$$\begin{aligned}\overline{S}^2{}_{x_1[f(x_2x_3)]} &= \sigma_z{}^2\left(\frac{n}{n-m}\right)\\ &= (0.0295)^2\left(\frac{120}{116}\right)\end{aligned}$$

$$S_{x_1[f(x_2x_3)]} = 0.0300$$

Similarly, the index of multiple correlation for X_1 as a joint function of X_2 and X_3 may be computed in the usual manner:

$$\begin{aligned}\overline{\mathrm{P}}^2 = 1 - \frac{\sigma_z{}^2}{\sigma_1{}^2} &= 1 - \frac{(0.0295)^2}{(0.1265)^2}\\ &= 1 - \frac{0.000870}{0.01600}\\ &= 0.9456,\end{aligned}$$

and adjusting for the number of constants

$$\begin{aligned}\overline{\mathrm{P}}^2 &= 1 - (1 - \mathrm{P}^2)\left(\frac{n-1}{n-m}\right)\\ &= 1 - (1 - 0.9456)\left(\frac{119}{116}\right)\\ &= 0.9442\\ \overline{\mathrm{P}} &= 0.972\end{aligned}$$

It is evident that the volume of a round haystack may be very closely estimated from the rough farm measurements of circumference and "over." The standard error of estimate, 0.0300, indicates that the logarithm of volume can be estimated to ± 0.0300 of the true logarithm

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[4] See page 311 for a further discussion of the possibilities of this type of formula.

$\frac{2}{3}$ of the time, and to ± 0.0600 of the true logarithm 95 per cent of the time. Taking the antilogarithms of 0.0300 and $\bar{1}.9700$, and of 0.0600 and $\bar{1}.9400$, we find that that means the volume can be estimated to between 107.2 per cent and 93.3 per cent, or between 114.9 per cent and 87.1 per cent, of the true values, respectively.

Stating the conclusions shown by the joint function.—After the joint relation of one variable to two others has been determined by the method sketched, the final regression surface, as expressed in Figures 58, 59, or 60, may be re-stated in simpler terms by preparing tables showing the average or expected values of X_1 for stated combinations of X_2 and X_3. In this particular problem, where the surface was determined with respect to logarithmic values, that involves determining the logarithms of X_2 and X_3 for the selected values, reading off from the charts the corresponding estimated value for the logarithm of X_1, and finding its antilogarithm. Carrying out this process, the values shown in Table 83 are obtained.

TABLE 83

AVERAGE VOLUME OF ROUND HAYSTACKS FOR DIFFERENT COMBINATIONS OF CIRCUMFERENCE AND "OVER."

Circumference	"Over," in feet				
	30	34	38	42	46
	Cubic feet	*Cubic feet*	*Cubic feet*	*Cubic feet*	*Cubic feet*
60 feet	1,730	2,244			
65 feet	1,871	2,432	3,097		
70 feet	1,928	2,553	3,319	4,150	
75 feet	....	2,655	3,524	4,467	5,623
80 feet	....	....	....	4,710	6,026

Determining Joint Influence of Two Independent Variables, Holding Other Independent Variables Constant.—In many cases it may be desirable to allow for the joint influence of two variables while simultaneously eliminating or holding constant the effect of one or more additional independent variables. In the case of the corn problem it would be desirable to determine the relation of yield to rainfall and temperature jointly, while simultaneously allowing for the upward tendency in yield during the period studied. This may be done by determining the relation according to the equation

$$X_1 = f(X_2, X_3) + f(X_4) \tag{76}$$

This relation may be worked out by combining the method just shown for determining a joint function for two independent variables, with the method of successive approximation for handling many variables, as discussed in Chapter 14. The essential steps are (1), to determine the curvilinear changes in X_1 with changes in X_2, X_3, and X_4, according to the simpler equation

$$X_1 = a + f(X_2) + f(X_3) + f(X_4)$$

then (2), compute the residuals for each observation, using these curves, and sub-classify the residuals according to the two variables for which the presence of a joint function is to be tested. If these averages of residuals indicate any significant warping of the surface, (3) they are next smoothed by the method presented following Table 81. The residuals may then (4) be adjusted to take account of this joint relation in addition to the individual curvilinear relations previously allowed for, and their standard deviation computed. If the variance has been significantly reduced, the residuals may then (5) be averaged with respect to the remaining independent factor, to see if the curve for that factor will be changed now that the joint relation to the other factors has been allowed for. If it is changed, the residuals are recomputed to see if any further change need be made in the joint function, and the process continued until the final shape of the curve and joint surface is determined.

Measuring Correlation with Respect to Joint Functions.—The correlation may be measured with respect to joint functions just as before it was measured with respect to curvilinear regressions. The standard error of the residuals, adjusted for the estimated number of constants, indicates the standard error of estimate; and this adjusted standard error, substituted in equation (66), gives the index of multiple correlation and of multiple determination. But since the *combined* influence of X_2 and X_3 is being determined, it is not possible to compute coefficients of partial correlation, or other measures of individual importance, for the variables which are being considered jointly. It would be possible to work out what portion of the variance in X_1 was accounted for by X_2 and X_3 together, and how much by X_4, but that would be all.[5] It is somewhat of a guess how many constants should be allowed for in computing the correlation and the standard error. It will be higher than for the individual curves for $f(X_2)$ and $f(X_3)$, in general. If the joint

[3] This would involve determining, by least squares, the equation

$$X_1 = a + b_2 f(X_2, X_3) + b_3 f(X_4)$$

The separate determination with respect to b_2 would then indicate the determination by X_2 and X_3 combined. See technical note 11.

relation merely involves a gradual regular shifting of the slope of the curves across the surface, one additional constant would be enough; if it involves a shifting at an increasing rate, two might be assumed; and if it involved several changes in shift, even more might be needed.

Determining Joint Influence of Three or More Independent Variables.—The methods just described might be used to determine several joint relations at the same time, each relation involving two independent variables. Thus if X_2 = rain in July, X_3 = temperature in July, X_4 = rain in August, X_5 = temperature in August, and X_6 = time, the yield of corn might be explained by a set of relations represented by the equation

$$X_1 = f(X_2, X_3) + f(X_4, X_5) + f(X_6) \tag{77}$$

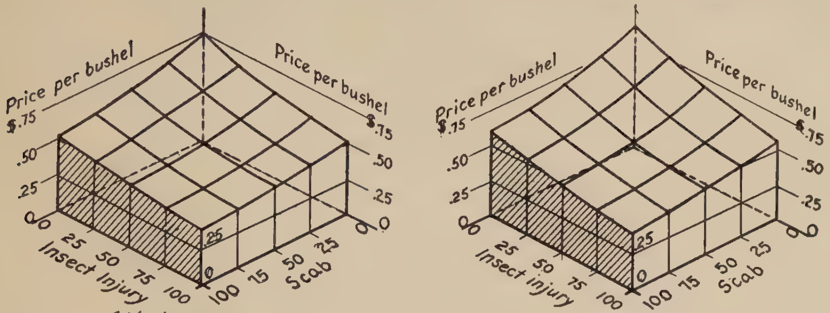
The functions would be determined by first getting the net regression curves for each factor separately, then the joint curves for (X_2, X_3) and (X_4, X_5) by classifying the residuals by the method just described, and then determining the final shapes by successive approximations. But there will be some cases where even so flexible a relation as represented in equation (77) will not be sufficient to really represent the relations. For example, yield might depend jointly on rainfall, temperature, and length of growing season, and a change in any one factor may cause differences in the effects of others as well. Such a relation would be represented by equations such as

$$X_1 = f(X_2, X_3, X_4, \ldots X_n) \tag{78}$$

To determine the shape of such a function for even 3 independent variables would require a large number of observations, since a threefold sub-classification would be needed. If only 4 classes were used for each variable, 64 sub-classes would be possible. Not unless there were sufficient observations so that say 3 to 5 might fall in each class, on the average, could such a relation be determined with any degree of accuracy, unless the correlation was very high indeed. If the correlation were perfect, one case to a sub-class might be sufficient to indicate the nature of the function.

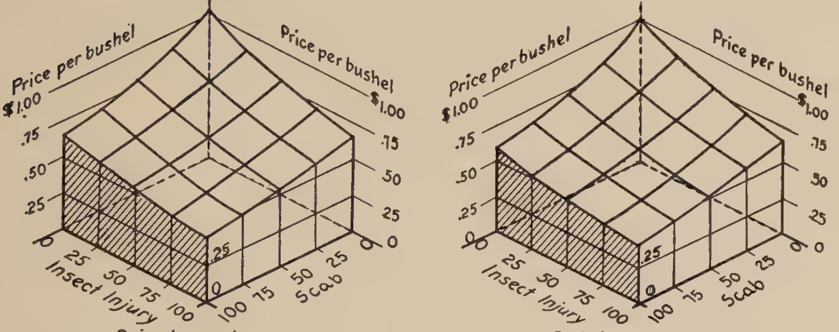
With three independent variables, successive smoothing in three dimensions would be involved. Where an adequate number of observations was available, the process might be simplified by dividing the observations into several groups according to one variable, determining the functional relation to the other two independent variables separately for each group, and then smoothing the results for the different groups together to determine the change in joint function with changes in the first variable.

Figure 61 illustrates some results of this sort, for a four-dimensional joint function. These results were obtained from an analysis of 190 observations of sales of individual lots of apples. The records were first separated into those for each of the 5 sizes of apples, and the joint functional relation of price to amount of insect injury and amount of scab determined separately for each size. These results were then smoothed between apples of different sizes, to make the "surface" of the imaginary four-dimensional solid diagram show a gradual continuous change over every dimension.[6]



2¼ inch apples

2½ inch apples



3 inch apples

3¼ inch apples

FIG. 61.—Average price of apples of given sizes, for various combinations of amount of insect injury and amount of scab.

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[6] This is done by reading expected prices for 0 scab, 0 insect injury, for each size of apple, and smoothing that series; reading for 0 scab, 20 per cent insect injury, and smoothing that series; and so on until every portion of the surface had been smoothed with respect to the third independent variable. The smoothed values could then be read off and smoothed again in other dimensions, until the final continuous function was obtained. This illustration is from an analysis supplied by Frederick V. Waugh.

While, of course, it is not possible to draw a single diagram expressing the four-dimensional relationship

$$X_1 = f(X_2, X_3, X_4)$$

the relation may be visualized by a composite diagram, as illustrated in Figure 61. This figure in particular illustrates the significant relations brought out by the joint functional treatment. Thus it is seen that large apples with neither scab nor insect injury, sell for a material premium over perfect apples of small size; but that if the apples are badly damaged it does not make much difference what size they are. This may be stated another way—the presence of defects reduces the price of large apples much more than the price of small ones. The figure also shows that the presence of either defect alone reduces the value of apples of any size materially, while the presence of both defects together reduces the price only slightly more. Thus for 3-inch apples, apples with 0 scab and 0 insect injury sold for \$1.05; those with 0 scab but 100 per cent insect injury for \$0.66; those with 100 per cent scab and 0 insect injury, for \$0.65; and those with 100 per cent scab and 100 per cent insect injury, for \$0.42. Increasing the insect injury from 0 to 100 per cent reduces the price 39 cents for apples with no scab, and only 23 cents for those with 100 per cent scab. Likewise for apples with neither scab nor insect injury 3-inch apples sold for \$1.05, and 2¼-inch ones for 75 cents; while for apples with 100 per cent of both injuries, the prices were 42 cents and 39 cents, respectively. These comparisons show what a difference the recognition of joint relations may make in research conclusions, and how important may be the resulting differences in the statement of relations.

Theoretically there is no limit to the number of variables which could be considered jointly. The only practical limitation is the number of observations available. Where it is possible to determine the joint relation, that affords by far the most satisfactory statement of the relationship, since then the real relation is not obscured by the assumptions hidden in the regression equation used. When a limited number of observations precludes a full recognition of joint relations, the use of mathematical transformations such as logarithms, logical grouping of the variables to determine the combinations of variables for which joint relations are most likely to obtain, and trial study of the residuals, will serve to make the final regression equation set forth the true nature of the relations as closely as possible from the limited evidence available.

Just as the accuracy of net regression curves depends largely on the number of observations along various portions of the curve, and the standard error of estimate, so the reliability of a regression surface (such as that shown in Figure 59) would depend on the standard error of estimate and the number of cases falling within the selected portion of the area.

Summary.—This chapter has developed means by which the relation between one variable and two others operating jointly may be determined, either where no other variables are concerned, or where one or more additional independent variables are taken into account. Methods are also discussed for measuring the influence of three or more independent variables operating jointly; but the increased number of observations necessary for such determinations restricts the field of usefulness of this type of analysis.

CHAPTER 21

SUPPLEMENTARY METHODS FOR DETERMINING CURVILINEAR AND JOINT RELATIONS

Chapters 14, 16, and 20 have set forth means by which curvilinear regressions may be determined for functions either of the simpler type

$$X_1 = f(X_2) + f(X_3) + f(X_4) + \ldots + f(X_n)$$

or of the more complex joint type

$$X_1 = f(X_2, X_3, \ldots, X_n)$$

In each case the methods were purely empirical, and depended on a combination of free-hand smoothing with successive approximation to the best curve as the influence of other factors was gradually eliminated. In addition to the methods which have been presented, there are other techniques which have been suggested for considering even more complex relations. On the other hand, if a specific mathematical function is assumed, the curves may be determined by a more rigid process, using the principle of "least squares." This chapter presents some of these further methods, both for multiple curvilinear regressions of the simpler forms, and for joint functional relations.

Determining Net Regression Curves by Mathematical Functions.—After the shape of the several net regression curves has been determined by the successive approximation method, a definite mathematical statement of the several functions may be obtained by an extension of the method presented on pages 188 and 189. The free-hand curves would provide the basis for selecting functions which would fit the net shape of the regression curves fairly well, giving at least this empirical criterion as to what function to use. Applying this method to the egg problem mentioned on page 242, for example, the final curves indicate that a straight line is probably adequate to describe the net regression of price on X_3; that a cubic parabola would probably be required to describe the net regression on X_2, and that a second-degree parabola might be sufficient to fit the net regression on X_4. Accordingly, the equation

$$X_1 = a + b_2X_2 + b_{2'}(X_2^2) + b_{2''}(X_2^3) + b_3X_3 + b_4X_4 + b_{4'}(X_4^2)$$

might be fitted to the data. After the values for the seven constants were determined by the usual method for linear correlation, the closeness with which the several mathematical curves fitted the net regressions could be judged by computing the residuals from the new regression equation, and then plotting them as deviations from the several net regression curves, exactly as the residuals from the linear regressions were plotted in Figures 34, 35, and 36. If modifications in the fitted curves were found necessary, they could be determined by the approximation process again.

Where the original relations indicate a marked curvilinear relation, as in Figures 33 and 41, the mathematical curves may be fitted right at the start, just as described above, and these curves used as the basis for subsequent corrections by the approximation method. Whether determining net linear regressions, as illustrated in the corn yield problem, or determining net curvilinear regressions, as just suggested, will prove the most expeditious way of beginning the successive approximation method, will depend on the circumstances in individual problems. Thus if one regression is known to be markedly curvilinear, while the others are substantially linear, taking that curvilinearity into account in the equation may bring the linear regressions for all the other variables much closer to their final form, and so reduce the number of steps necessary in the successive approximation determinations.

Determining the curves by least squares.—The process of determining net regression curves by the use of a definite mathematical equation may be illustrated for the following data:

X_2	X_3	X_1	X_2	X_3	X_1
1	3	8	0	2	7
2	2	10	4	5	9
4	7	8	3	3	10
9	8	9	1	2	9
5	5	10	6	5	10
2	3	9	1	2	9
2	2	9	2	2	10
7	14	7	4	14	6
9	8	9	1	2	9
2	4	8	10	11	8

Preliminary examination by the graphic method indicates that the net regressions of X_1 on both X_2 and X_3 may be approximately represented by parabolas. Accordingly, a net regression curve may be assumed of the type

$$X_1 = a + b_2X_2 + b'_2(X_2^2) + b_3X_3 + b'_3(X_3^2)$$

The arithmetic required to determine the five constants can be reduced by " coding " the squared values. Let $U = X_2^2/10$, and $V = X_3^2/10$. The regression equation then may be written

$$X_1 = a + b_2X_2 + b_uU + b_3X_3 + b_vV$$

The normal equations to determine the four constants are next obtained in exactly the same manner as described in Chapter 12 for a multiple correlation involving four variables. The resulting normal equations are as follows:

$$\begin{aligned}
(\Sigma x_2^2)b_2 + (\Sigma x_2u)b_u + (\Sigma x_2x_3)b_3 + (\Sigma x_2v)b_v &= \Sigma x_1x_2\\
(\Sigma x_2u)b_2 + (\Sigma u^2)b_u + \Sigma(x_3u)b_3 + (\Sigma uv)b_v &= \Sigma x_1u\\
(\Sigma x_2x_3)b_2 + (\Sigma ux_3)b_u + \Sigma(x_3^2)b_3 + (\Sigma x_3v)b_v &= \Sigma x_1x_3\\
(\Sigma x_2v)b_2 + (\Sigma uv)b_u + \Sigma(x_3v)b_3 + (\Sigma v^2)b_v &= \Sigma x_1v
\end{aligned}$$

Carrying out the required computations, the equations are found to be

$$\begin{aligned}
171.750b_2 + 170.625b_u + 165.000b_3 + 207.600b_v &= -\ \ 2.50\\
170.625b_2 + 181.165b_u + 153.540b_3 + 192.316b_v &= -\ \ 5.31\\
165.000b_2 + 153.540b_u + 295.200b_3 + 441.480b_v &= -\ 50.80\\
207.600b_2 + 192.316b_u + 441.480b_3 + 696.072b_v &= -\ 86.52
\end{aligned}$$

The $(\Sigma x_1^2) = 24.20$.

Solving these equations by the usual method and computing a from equation (41) by restating it

$$a_{1.2u3v} = M_1 - b_2M_2 - b_uM_u - b_3M_3 - b_vM_v$$

the regression equation is found to be

$$X_1 = 9.411 + 1.2709\,X_2 - 0.7337\,U - 0.9957\,X_3 + 0.3309V$$

The net regressions of X_1 on X_2 and X_3 are now shown by the two parabolic equations:

$$X_1 = 5.596 + 1.2709\,X_2 - 0.7337\,X_2^2$$

and

$$X_1 = 12.515 - 0.9957\,X_3 + 0.03309\,X_3^2$$

The graph of these two curves is shown in Figure 62.

The (unadjusted) multiple correlation of X_1 with X_2, U, X_3, and V

is 0.968. Since there are five constants in the regression equation, and 20 observations, this indicates (by equation 47) a probable true correlation of 0.963. This is then the index of multiple correlation of X_1 with X_2 and X_3, according to the parabolic regressions. The standard error of estimate, similarly adjusted, is found to be 0.101.

It should be noted that where net curvilinear regressions are found by this method, the number of constants assumed in the regression equation is definitely known, and there can be no question as to the exact correction to apply to the computed correlation and standard error, or as to the probable significance of the observed correlation. On the other hand, the shape of the regression curve obtained is conditioned by the type of curve assumed; except where there is some logical basis for assuming a particular type of relation, the selection of the formula is still a purely empirical process. If the formula selected does not fit the data well, the resulting curves may fail to reveal the true relations.

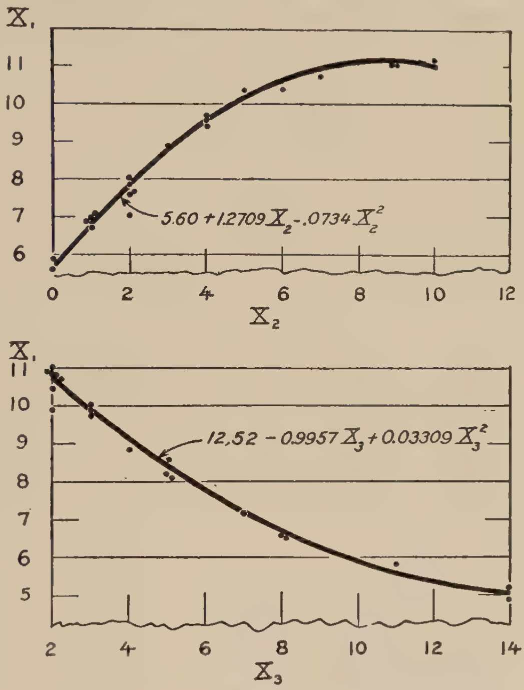

FIG. 62.—Parabolic regression curves, fitted simultaneously, and net residuals.

Testing the Fit of the Curves Graphically.—The extent to which mathematical net regressions fail to adequately fit the data may be investigated, in any particular problem, by the same graphic methods set forth in Chapter 14. To make this check, after the regression equation is determined for the particular curves selected, estimated values of X_1 are calculated from the equation. The residual differences between X_1 and these estimated values are then computed. These residuals are then plotted as departures from the mathematical net regression curves, in the same manner that the residuals from linear regressions were plotted as departures from the linear regressions in Chapter 14. Carrying this out for the problem illustrated, the results are as follows:

X_2	X_3	X_1	X'_1	z	X_2	X_3	X_1	X'_1	z
1	3	8	7.9	0.1	0	2	7	6.7	0.3
2	2	10	9.8	0.2	4	5	9	9.2	−0.2
4	7	8	8.0	0.0	3	3	10	9.9	0.1
9	8	9	9.1	−0.1	1	2	9	8.8	0.2
5	5	10	9.8	0.2	6	5	10	10.2	−0.2
2	3	9	9.0	0.0	1	2	9	8.8	0.2
2	2	9	9.8	−0.8	2	2	10	9.8	0.2
7	14	7	7.2	−0.2	4	14	6	5.9	0.1
9	8	9	9.1	−0.1	1	2	9	8.8	0.2
2	4	8	8.2	−0.2	10	11	8	7.8	0.2

The residuals obtained above are then plotted as departures from the parabolic net regressions, as also shown in Figure 62. It is evident in this case that the parabolic regressions represent the relations quite well, with the departures in general evenly distributed on both sides of each curve throughout their length. Only in the curve for $X_1 = f(X_2)$ is there any indication of failure to obtain a good fit. Here most of the individual observations lie slightly above the curve for values of X_2 below 2. Above $X_2 = 8$ the individual observations do not agree with the downward turn of the parabola. Using a third degree parabola for $f(X_2)$, which would mean adding a new term, $b''_2(X_2^3)$, to the regression equation, would produce a better fit for this function.

Where the graphic check on the adequacy of a mathematical net regression curve indicated that the functional relation was such that it could not be readily represented by a higher order parabola or other simple mathematical expression, a free-hand curve might be fitted to the residuals instead, and the final shape of the curves determined by successive approximations, just as described in Chapter 14. The determination of parabolic net regressions may thus be substituted for the determination of linear net regressions as the first step in the successive approximation method of obtaining net regression curves.

Any other type of mathematical function, the parameters of which can be expressed in the first degree, can be used to determine net regressions by the method of least squares. Besides still higher powers of X_2, such transformations as $10/X_2^2$, $100/X_2^3$, $\log X_2$, and $1/\log X_2$ may be employed as independent factors, either in place of the previous independent factors or as an addition to the simple statement of them.

Supplementary Methods of Determining the Final Shape of Net Regression Curves.—After the shapes of the several regression curves have been determined by the method of successive approximations, it is sometimes desirable to use the method of linear correlation to determine whether any further adjustment should be made in the *slope* of the several curves, to give the closest possible estimate of the X_1 values. There are two alternative ways of doing this, yielding slightly different types of corrections. The first and simplest method is to correlate the final residuals, z'''', with the values of the several independent variables. That is, a new multiple correlation is run to determine the regression equation

$$z'''' = a_{z.234} + b_{z2.34}X_2 + b_{z3.24}X_3 + b_{z4.23}X_4 \tag{79}$$

If significant values are obtained for any of the b's, they indicate that the corresponding regression curve should be rotated counter-clockwise, if the net regression coefficient is positive; or clockwise if the coefficient is negative. The final values of the several functions will then equal the readings for the curves as previously determined, plus the additional linear correction. That is, if the final curvilinear multiple regression equation is to be

$$X_1 = k + f_2(X_2) + f_3(X_3) + f_4(X_4)$$

the several terms will be as follows:

$$\begin{aligned} k &= a''''_{1.234} + a_{z.234} \\ f_2(X_2) &= f'''_2(X_2) + b_{z2.34}X_2 \\ f_3(X_3) &= f'''_3(X_3) + b_{z3.24}X_3 \\ f_4(X_4) &= f'''_4(X_4) + b_{z4.23}X_4 \end{aligned}$$

Since the intercorrelations between X_2, X_3, and X_4 have already been computed in determining the original linear net regressions, much of the work required in determining the constants for equation (79) has already been performed, and the additional computation involved is not very heavy.

A somewhat different type of correction is obtained by determining the regression equation

$$\begin{aligned} X_1 = a_{1.2'3'4'} &+ b_{12'.3'4'}\,[f'''_2(X_2)] + b_{13'.2'4'}\,[f'''_3(X_3)] \\ &+ b_{14'.2'3'}\,[f'''_4(X_4)] \end{aligned} \tag{80}$$

To compute the new constants required in equation (80), the functional readings corresponding to the independent variables are correlated with the original values of the dependent variable. Thus, if the values in

Table 64, page 210, had been obtained from the final curves determined by the successive approximation process, the values read from the curves, shown in the 4th, 5th, and 6th columns, would have been substituted for the original independent variables in running the multiple correlation with X_1. If we use X'_2, etc., to represent these transformed values, the data to be correlated, for the first four sets of observations would be:

X'_2	X'_3	X'_4	X_1
7.4	11.7	12.3	24.5
7.9	13.0	11.8	33.7
8.4	12.2	12.2	27.9
8.8	9.9	12.2	27.5

If the net regression coefficients come out 1.0, in equation (80), that indicates that no change need be made in the curves. If any b comes out other than unity, however, the values read from the corresponding curve should be adjusted as indicated by the regression results. The adjustment may be worked out as follows:

In the same way that $f'''_2(X_2)$ was used to indicate the values read from the final set of approximation curves, let $f'''_2(x_2)$ represent the deviations of those readings for each variable from the average of all the readings for the particular variable. That is, for each observation

$$f'''_2(x_2) = f'''_2(X_2) - M_{f'''(X_2)}$$

The regression equation (80) may then be restated

$$x_1 = b_{12'.3'4'}[f'''_2(x_2)] + b_{13'.2'4'}[f'''_3(x_3)] + b_{14'.2'3'}[f'''_4(x_4)]$$

and the corrected functions will be as follows:

$$f(X_2) = f'''(X_2) + b_{12'.3'4'}[f'''(x_2)]$$

$$f(X_3) = f'''(X_3) + b_{13'.2'4'}[f'''(x_3)]$$

$$f(X_4) = f'''(X_4) + b_{14'.2'3'}[f'''(x_4)]$$

The difference between the two types of corrections is illustrated in Figure 63. Here the final curve for $f(X_4)$, from the corn yield problem, has been plotted, and in addition it is shown as if a correction $+ 0.5x_4$ had been worked out by the first method, equation (79), or a correction $1.5[F(X_4)]$ had been determined, by the second method, equation (80). It is evident that the first correction rotates the curve, so as to make its upward slope greater throughout, and its downward slope less; whereas the second correction merely expands the curve, making all the high values higher, and all the low values lower, no matter where they fall with respect to X_4. This is typical of the effect of these two types of corrections when applied to a curve of the type shown here. For curves which do not depart so far from a straight line, and which either rise or fall through their entire length, the difference between the two types of correction is less marked, as may readily be determined by experiment. For a straight line the correction given by the two methods will tend to be identical.

The Bruce adjustment.—Besides the two methods shown of adjusting the final curves by linear correlation (the method of least squares), there is a somewhat different adjustment of the final readings, termed the Bruce method after its originator.[1] This method consists essentially of (1) constructing a dot-chart showing the relation between the original values of X_1 and the values, X'''_1, estimated from the final set of curves (even after corrections such as those just mentioned have been applied); (2) drawing in a curve showing average values of X_1 for corresponding values of X'''_1; and (3) using this curve as the basis for making the final estimate. This method thus consists in finding the function θ for the equation

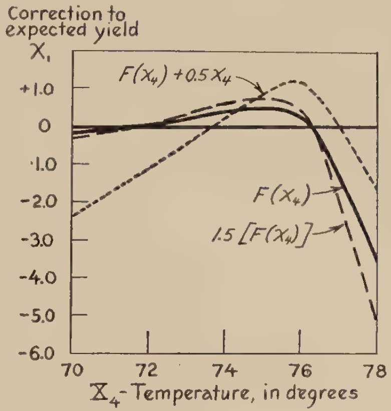

FIG. 63.—Two types of corrections to net regression curves.

$$X''''_1 = \theta(X'''_1) \tag{81}$$

$$X''''_1 = \theta[a + f'''(X_2) + f'''(X_3) + f'''(X_4)] \tag{82}$$

The function θ corresponds to the curve just described.

The curve for θ can be determined only after all the other f's are worked out. The average value of X_1 is determined for each group of estimated values, X'''_1, treating X'''_1 as if it were a single independent variable. Plotting the average values of both variables against each other, just as in Figure 23, a curve may be fitted free-hand if it is indicated, to show the change in X_1 with changes in X'''_1. This curve is then the function θ for equation (81).

This function may then be used to work out new estimates, X''''_1, which should have still higher correlation with X_1 than the previous estimates.

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[1] See list of references at end of this chapter.

The Bruce adjustment is likely to prove of value in certain types of joint functions. Thus in the egg price problem, it might be that when values of the several factors which by themselves each tend to lower the price, all occur in combination, the resulting price would be, on the average, even lower than the sum of the effects of each of the variables would indicate. On the other hand, it might be that when values of several factors, each of which would raise the price, occurred together, the price would not go quite as high as the sum of the probable effects of the several factors would indicate. The Bruce adjustment thus makes it possible to determine one type of joint relation without the considerable extra work described in Chapter 20 for determining joint functions.

Determining Joint Relations by Contours.—The method for determining joint relations presented in the last chapter is essentially

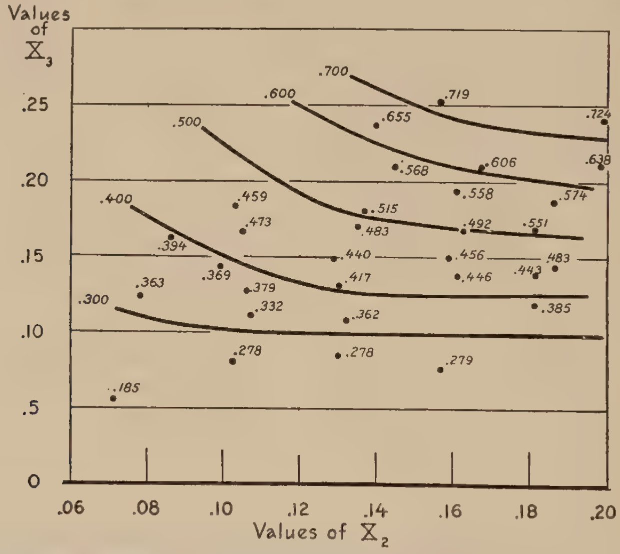

FIG. 64.—Average values of X_1 for various combinations of X_2 and X_3, and contours fitted directly to the data.

one of sub-classification and then two-way smoothing of the resulting averages, by successive smoothing for each of the two (or more) independent variables. A somewhat different method has been worked out by which the three-variable surface may be smoothed directly in both independent dimensions at the same time. The Waugh method is based on determining contours directly, instead of indirectly as described in the last chapter. In using this method, the averages of sub-groups (either of original observations, as in Table 81, or of residuals) are plotted directly on a two-variable diagram, with one independent variable as ordinate and the other independent variable as abscissa, and using the group averages for the two independent variables. The average of the dependent variable (or residual) is then written in next to the dot which designates the sub-group. Figure 64 shows such a chart for the data of Table 81. The next step is to connect averages of equal values by a continuous line, or, if none are the same, to run in contour lines which will enclose averages within the same limits. Thus the lines on the chart have been drawn so as to separate off the groups with X_1 under 0.300, between 0.300 and 0.400, from 0.400 to 0.500, etc. Once the shape and direction of these contours are determined, they may then be redrawn so as to keep a similar shape or a continuously-changing shape, and an even or a regularly-changing interval, across the whole surface. It is evident that Figure 64 is quite similar to Figure 60, determined by the other method.

Where the correlation is high, so that the individual observations define the regression surface rather closely, the Waugh method may be used directly with the individual observations, plotting each observation in the same way that the group averages were plotted in Figure 64. The following data illustrate this use of the method.

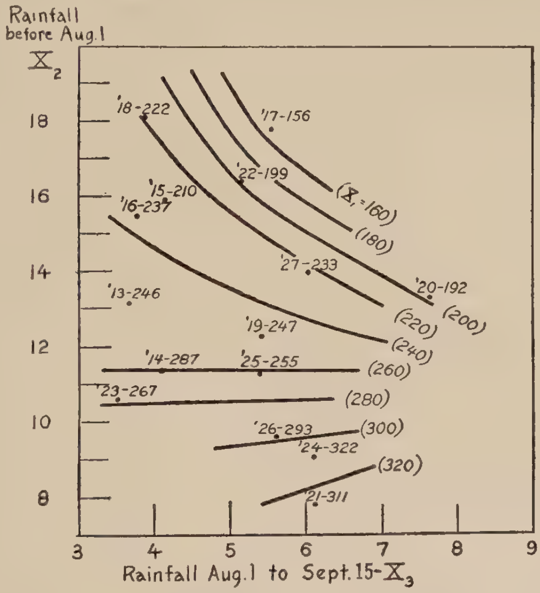

Fig. 65.—Yield of potatoes for years of specified rainfall before August 1 and after August 1, and contours fitted directly to the data.

The data from Table 84 are plotted in Figure 65, using the deviation of yield from trend as the dependent factor. Drawing in con-

TABLE 84

WEATHER CONDITIONS AND YIELD OF POTATOES IN MAINE

Year	Rainfall to August 1 (July doubled)	Rainfall August 1 to September 15	Yield	Adjustment for trend *	Yield adjusted for trend	Estimated yield	Residual
	Inches X_2	*Inches* X_3	*Bushels* X_1	*Bushels*	*Bushels* X_1	*Bushels* $f(x_2x_3)$	z
1913	13.17	3.66	220	+26	246	248	− 2
1914	11.33	4.08	260	+27	287	260	27
1915	15.96	4.12	179	+31	210	229	−19
1916	15.46	3.77	204	+33	237	236	1
1917	17.77	5.53	125	+31	156	155	1
1918	18.09	3.87	200	+22	222	220	2
1919	12.25	5.41	230	+17	247	248	− 1
1920	13.29	7.62	177	+15	192	196	− 4
1921	7.82	6.11	298	+13	311	323	−12
1922	16.40	5.12	187	+12	199	197	2
1923	10.61	3.51	258	+ 9	267	278	−11
1924	9.10	6.13	315	+ 7	322	308	14
1925	11.30	5.38	250	+ 5	255	262	− 7
1926	9.60	5.60	290	+ 3	293	297	− 4
1927	13.98	6.02	232	+ 1	233	226	7
1928	15.45	6.45	220	− 1	219		

* Simultaneously determined while allowing for trend. See F. A. Waugh, "Methods of Forecasting New England Potato Yields," U. S. Dept. of Agriculture, Bureau of Agricultural Economics, Mimeographed Report, Feb., 1929.

tours so as to separate years of similar yields, it is found that a very peculiar type of surface is indicated—one that changes elevation very rapidly between the combination of high early rainfall and low late rainfall, and high early rainfall and high late rainfall. When these results are used to forecast the yield in 1928 (which year, it will be noted, was not plotted or used in determining the contours) a yield of about 175 bushels is indicated. This is only in fair agreement with the final yield of 219 bushels, determined several months after the climatic data were available to give the forecast stated.

Reading off the estimated values for each year shown, the estimated adjusted yields as shown in the next to the last column of Table 84 are obtained. The standard deviation of the residuals, shown in the next column, is 10.6 bushels, while the σ of the deviations from trend is 63.0. Assuming five constants to be necessary to represent the surface mathematically, the standard error of estimate would be 13.0 bushels, and the index of correlation for the surface indicated by the contours would be 0.98. Assuming that the trend line could be fairly accurately projected, the standard error of estimate indicates that an error as great as that in 1928 would be likely to occur only very rarely. The fact of high correlation and of low standard error could be judged directly from the closeness with which the contours fit the individual observations, in just the same way that closeness of the observations to the regression line indicates high correlation in the case of simple correlation.

Determining Joint Functions by Definite Mathematical Functions.—In exactly the same way that definite equations can be determined by the method of least squares to represent curvilinear net regressions, certain types of joint functional surfaces can be represented by definite equations. The simplest type is that shown by the haystack volume problem in Chapter 20, where the regression of X_1 on X_3 is substantially linear for any given value of X_2, but where the slope of the regression $b_{13\cdot2}$ changes as the values of X_2 change. If it is assumed that the slope of $b_{13\cdot2}$ changes at a constant rate with changes in X_2, this assumption may be expressed in the relation

$$X_1 = a + b(c + dX_2)X_3$$

Multiplying this out, it becomes

$$X_1 = a + bcX_3 + bdX_2X_3$$

Which may be stated

$$X_1 = a + eX_3 + g(X_2X_3) \qquad (83)$$

The values of a, e, and g may then be determined by the usual methods of linear multiple correlation, using X_3 and (X_2X_3) as the independent factors.

If it is assumed that X_1 varies with X_2, other than through its influence on $b_{13\cdot2}$, an additional term may be added to the equation, making it

$$X_1 = a + eX_3 + g(X_2X_3) + hX_2 \qquad (84)$$

Determining the values of the four constants of equation (84) from the haystack data, and working out estimated values of X_1 for specific combinations of values of X_2 and X_3, the same joint functional surface will be arrived at as was determined by the graphic method presented in Chapter 20.

The same method may be extended to n independent variables, assuming similar linear net regressions for X_1 on each independent factor, with the other independent factors constant at any given values;

and changing their slope progressively and uniformly as the other independent factors change. For three independent factors (four-dimensional space) the regression equation would be

$$X_1 = a + b_2X_2 + b_3X_3 + b_4X_4 + c_2(X_2X_3)$$
$$+ c_3(X_2X_4) + c_4(X_3X_4)$$

Determination of the seven constants would thus make possible a definite mathematical representation of a very complex set of relationships.

If it is assumed (1) that the regression of X_2 on X_1, for any given value of X_3, is a curve, and (2) that the slope of this curve changes *at a changing rate* as X_3 changes, this assumption may be stated

$$X_1 = a + f\,[a + \theta(X_3)]X_2$$

This equation may be approximately represented by the following form:

$$X_1 = a + f(X_2) + f(X_2X_3) + f(X_4) \tag{85}$$

Using X_2, X_3, and the product (X_2X_3) as the independent factors, the shape of the three functions may be determined by any of the methods presented previously. Then working out estimated values of X_1 for various combinations of X_2 and X_3, very warped curvilinear surfaces for $f(X_2, X_3)$ can be determined. This last method is extremely flexible, and can be used to determine a wide variety of joint functional relations. It, too, may be generalized for n variables, with increasing numbers of observations. For three independent variables it would be

$$X_1 = a + f(X_2) + f(X_3) + f(X_4) + f(X_2X_3) + f(X_2X_4) + f(X_3X_4)$$

Although these methods do not reduce greatly the number of observations required to determine joint functions, they do make it possible to apply the systematic procedure developed in Chapter 14; to judge more accurately the number of constants represented in the regression surface; and they enable the methods of Chapter 19 and technical note 13 to be applied in judging the reliability of the conclusions.

The Court method.—The Waugh method is essentially a way of simplifying the smoothing of the surface, while still leaving it primarily a graphic free-hand process. Another method, developed by Andrew Court, reduces the determination of joint functions to a more definite process, similar to the determination of the usual regression curves.

This method depends upon a mathematical rotation of the surface of cubes such as those shown in Figures 53 to 55, so that instead of averaging the values only when viewed with respect to the rectangular axes X_2 and X_3, they may also be averaged with respect to axes cutting across the surface at an angle. Though similar to the mathematical method just described, this method is applicable to a somewhat different type of surface.

The characteristic feature of the method is the use of composite functions which represent two or more independent variables. Thus the regression surface in Figure 61, for apples of one size, might be expressed by the equation

$$X_1 = f(X_2) + f(X_3) + f(X_2 + X_3) \tag{86}$$

The effect of the introduction of the new composite element, $(X_2 + X_3)$, may be explained by working out what the values of this composite variable will be for various combinations of X_2 and X_3. The following statement shows this in detail.

Value of Composite Variable $(X_2 + X_3)$, for Various Values of X_2 and X_3

X_3 values	X_2 values					
	0	20	40	60	80	100
0	0	20	40	60	80	100
20	20	40	60	80	100	120
40	40	60	80	100	120	140
60	60	80	100	120	140	160
80	80	100	120	140	160	180
100	100	120	140	160	180	200

It will be seen that the composite values $(X_2 + X_3)$ run diagonally across the surface. Thus the value 100 occurs with $X_3 = 100$, $X_2 = 0$; with $X_2 = 50$, $X_3 = 50$; and with $X_2 = 0$, $X_3 = 100$. If the surface shown in Figure 61 were to be described by equation (86), the curve for $f(X_2 + X_3)$ would rise gradually from 0 to 100, then rise more and more sharply as it approached 200.

One advantage of the Court method is that it makes it possible to estimate with much greater accuracy the number of constants represented by the surface. Thus in Figure 61 each curve might reasonably be represented by a 2nd-degree parabola, so 7 constants may be assumed for the entire relation in equation (86). If desired, the equation could be written in terms of parabolas, as follows:

$$X_1 = a + b_2X_2 + b'_2(X_2{}^2) + b_3X_2 + b'_3(X_3{}^2)$$
$$+ b_4(X_2 + X_3) + b'_4(X_2 + X_3)^2$$

Stated in this form, the shape of the surface could be determined by a least squares solution, giving exactly determinable shapes for each of the three functions, and a definite measure of the reliability of the results. However, unless the mathematical curves happened to be exactly right to represent the real relations, the final functions might not express the true relationship so accurately as would the free-hand curves, determined by successive approximations.

Where two independent variables which are to be considered jointly are not of the same degree of variability, a 45-degree rotation of the surface, such as that shown in the tabular statement of $(X_2 + X_3)$, could still be secured by making the composite variable equal to $\frac{X_2}{\sigma_2} + \frac{X_3}{\sigma_3}$. If a negative rotation were desired, that could be obtained by using the value $\frac{X_2}{\sigma_2} - \frac{X_3}{\sigma_3}$. Further, if it were desired to rotate the surface either less or more than 45 degrees, that could be done by dividing one variable or the other by a suitable constant. Thus the form $\frac{X_2}{\sigma_2} + 2\frac{X_3}{\sigma_3}$ would rotate the surface about 67 degrees.

The general statement for the Court solution, for a two-variable joint function, is as follows:

$$X_1 = f(X_2) + f(X_3) + f\left(\frac{X_2}{a} + \frac{X_3}{b}\right) + f\left(\frac{X_2}{a} - \frac{X_3}{c}\right) \qquad (87)$$

Even a more complex form than equation (87) could be employed, by using combination functions of several different degrees of rotation in the same equation. Using such a combination with simple parabolas for each function, Court has successfully fitted the regression surface shown in Figure 54, illustrating the flexibility of the method. It is evident, however, that much judgment is necessary in selecting the way the combination variable or variables are to be stated in equation (87), both with respect to whether the rotation is to be positive, or negative, or both, and the extent of the rotation to be used. In the apple price problem, it was known that the statement of equation (86) would fit quite well, because of the prior knowledge of the relations expressed in Figure 61. Where such information as to the shape of the function is not known *a priori*, considerable testing of different methods of statement and examination of group averages and profile charts like Figure 56 would be necessary to decide upon a form of statement which would yield adequate results.

The Court method may be extended to *n*-dimension joint functions, and has very great flexibility for this purpose. The number of possible combination variables becomes increasingly great as the number of variables increases, however, so that stable results cannot be secured by this method either, unless a sufficiently large number of observations is available to define the relations for each of the subclasses which are obtained by successive sorting on each independent variable. Thus using only 45-degree rotations, the full Court equation for three independent variables would be

$$\left.\begin{aligned} X_1 = f(X_2) + f(X_3) + f(X_4) \\ + f\left(\frac{X_2}{\sigma_2} + \frac{X_3}{\sigma_3} + \frac{X_4}{\sigma_4}\right) + f\left(\frac{X_2}{\sigma_2} + \frac{X_3}{\sigma_3} - \frac{X_4}{\sigma_4}\right) \\ + f\left(\frac{X_2}{\sigma_2} - \frac{X_3}{\sigma_3} + \frac{X_4}{\sigma_4}\right) + f\left(\frac{X_2}{\sigma_2} - \frac{X_3}{\sigma_3} - \frac{X_4}{\sigma_4}\right) \end{aligned}\right\} \quad (88)$$

Even if each function were represented by only two constants, equation (88) would involve fourteen constants. The similar forms for four and five independent variables become increasingly complex. It is probable that this method can be used only occasionally, where a very large number of observations can be obtained. For such problems, however—as for the apple price example, where 190 observations were available—the use of equations such as (87) or (88) might reduce somewhat the factor of individual judgment, and enable the researcher to determine joint relations in *n*-dimensional space with more facility than by graphic methods which essentially involve considering individual dimensions in succession, or at most two at the same time.

Measures of Correlation for Mathematically-determined Regressions.—Where the curvilinear net regressions or regression surfaces have been determined by the use of mathematical functions such as those indicated in equations (55) and (56), then the several measures of closeness of fit can be obtained from the computations employed in determining the values of the several *b*'s by the usual linear multiple correlation methods. For example, if equation (56) involving cubic parabolas for each variable has been employed, the regression equation is

$$X_1 = a + b_2X_2 + b_{2'}(X_2)^2 + b_{2''}(X_2)^3$$
$$+ b_3X_3 + b_{3'}(X_3)^2 + b_{3''}(X_3)^3 + \text{(etc.)}$$

In that case the *coefficient* of multiple correlation with respect to the independent variables X_2, X_2^2, X_2^3, X_3, X_3^2, X_3^3, etc., becomes the *index* of multiple correlation with respect to the variables X_2, X_3, etc. The necessary adjustments because of the number of constants represented in the regression equation, as indicated by equation (67), still have to be made, of course; if the regression curves have been determined mathematically, there is no question as to the value of m to be used. For the equation shown above, with only two independent variables, X_2 and X_3, m is 7. With this limitation, the index of multiple correlation for mathematically determined regressions may be defined by the equation

$$R_{1.2,\, 2^2,\, 2^3,\, 3,\, 3^2,\, 3^3,\, \ldots\, n,\, n^2,\, n^3} = \mathrm{P}_{1.23\ldots n} \tag{89}$$

Indexes of partial correlation and of part correlation could be worked out by parallel recombination of the elements involved in determining the constants of equation (56), but the steps necessary would become exceedingly complicated, and therefore are not set forth here.

The standard error of estimate in using a mathematically determined curvilinear regression equation is the same as the standard error of the multiple correlation results, with the appropriate correction for the number of constants. When the index of multiple correlation has been determined, with the proper adjustments, the standard error of estimate may be readily obtained by the formula

$$\bar{S}^2_{1.f(23\ldots n)} = \sigma_1^2(1 - \bar{\mathrm{P}}^2_{1.23\ldots n})\left(\frac{n}{n-1}\right) \tag{90}$$

This operation is necessarily identical with that employed in computing the standard error in linear multiple correlation, using the adjusted coefficient of multiple correlation.

Summary.—Simple curvilinear regressions, determined by the successive approximation process, may be subjected to a final correction by mathematical means; or mathematical curves for each function may be fitted simultaneously; or certain types of joint relations may be represented by the use of a composite function, θ, which may be determined rather readily.

The smoothing of two-variable joint functions may be facilitated by the use of contours (the Waugh method) drawn free-hand either from the sub-group averages, or, in the case of high correlation, from the original observations. Other methods employ combination variables composed of simple linear functions of two or more independent variables to rotate or warp the joint plane and so determine its shape other than at right-angles to the axes of the independent variables. By using several such combination variables, and determining regression curves for them by successive approximation, very complex joint functional surfaces may be represented quite closely. These methods may be extended to joint functions of n variables, but they become increasingly complex and require an increasingly large number of observations. Even so, these methods reduce somewhat the element of human judgment involved in the determination of joint functions, and simplify the steps involved to more nearly a routine process which can be expected to give identical results from the same data in the hands of different investigators.

It is possible to obtain standard errors of estimate, and indexes of multiple correlation, which serve the same purpose for curvilinear multiple regression equations that the comparable coefficients serve for linear multiple regressions. Owing to the extent to which the process of fitting the curves may exaggerate the significance of the results, it is even more important to adjust the several measures with respect to the number of observations and number of constants involved than it is with linear multiple correlation.

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CHAPTER 22

TYPES OF PROBLEMS TO WHICH CORRELATION ANALYSIS HAS BEEN APPLIED

In the preceding chapters many different practical problems have been used to illustrate the kinds of correlation analysis and the actual steps in working out the results. It may now be worth while to turn attention to specific research problems to which these methods have been applied in the past. This will indicate the type of logical analysis which must be made before the statistical technique can be applied, and show something of the kind of conclusions which may be reached by the use of these techniques.

Land Values.—One of the first comprehensive studies involving extensive correlation analysis was a study of land values by Haas (1).[1] In this study the sales prices of a number of different farms were obtained, and also supplementary facts about the farms, such as distance from town, value of buildings, proportion of crop land, fertility of the soil, and type of road on which the farm fronted. Changes in land values over the period were first eliminated, and the adjusted acre-prices related to the other factors by linear correlation. Sortings of the residual values were used to determine the regressions for some of the less important factors. It was found that the differences in value per acre had a multiple correlation of $R = 0.81$ with the factors mentioned and that acre values could be estimated from the independent factors with a standard error of $19 per acre. As the assessor's valuations of these same farms showed a much larger error, as compared with the actual sales values, it was suggested that the impartial regression equation be substituted for the less reliable human judgment in assessing individual farms for taxation purposes.

In a later study of the same type (2) the value of the farm dwelling and of the barns were considered as separate variables, and curvilinear regressions were determined. It was found in this study that the contribution of the farm dwelling to the farm value was a joint function of the value of the dwelling and the size of the farm, an expensive dwelling adding more to the value of a large farm than to the value of a small one. Road type was one of the factors considered. Three classes of roads were used, and the method explained in Chapter 17 was employed to determine the net difference in farm value per acre with differences in the type of road. Preliminary work in this study, with the farm value stated on a per-acre basis, gave a linear correlation of $R = 0.98$. It was discovered, however, that this high correlation was due almost entirely to the presence of a few very small farms, which showed values of farms per acre and values of buildings per acre both running into the thousands of dollars. When these farms were excluded, the linear multiple correlation dropped to $R = 0.64$, indicating the spurious correlation obtained by dividing by the common factor, number of acres. In the final correlation, using curvilinear relations and joint functions, a multiple correlation of $\mathrm{P} = 0.77$ was obtained. As 368 observations were used, more complex methods could be employed for this analysis than would be feasible in most cases.

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[1] The numbers in parentheses refer to references at the end of this chapter.

Physical Relations Between Input and Output.—Another type of problem to which multiple correlation has been applied is to determine the physical relation between the number of input (or cost) elements applied in some production process, and the resulting output or yield. This problem is particularly important in agricultural research, where many of the combinations of conditions which occur in practical farming cannot be reproduced or studied under experimental conditions; and where the number of variables is so great as to make the use of fully controlled experiments both lengthy and costly.

In one of these studies (3) the gain in weight of beef steers on feed was related to the quantities of corn, hay, and high-protein feeds fed per day, to the number of days on feed, and to the initial weight of the animals. Curvilinear regressions were determined for all factors, protein being the only one to indicate a true linear relation. The curves showed marked diminishing returns per unit of feed as added amounts of corn or of hay were fed per day. The younger the animals, and the shorter the time they were on feed, the smaller the amount of feed necessary to produce a given amount of gain. There were 67 observations for this study, each representing a different bunch of cattle. The multiple correlation was $\mathrm{P} = 0.78$.[2]

The same analysis of physical relations has been applied to the production of milk by dairy cows. In most of these studies (4, 5, 6, 7) the feeds used and milk produced have been worked out on a herd-average basis, the record for each herd constituting one observation. In one study, however, the records were available by individual cows, and the conclusions secured from those records agreed quite well with those obtained from the other analyses (8).

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[2] This study is of particular interest to the author, as it was in studying this particular problem that the successive approximation method of determining net regression curves was first worked out, and in this problem that regression curves were first determined by this method while holding the influence of other variables constant.

The total quantity of digestible nutrients in the feed; the proportion of protein in the feed; the proportion of butter-fat in the milk; and the proportion of the herd freshening in the fall, have all been found to be important variables influencing the production of milk. Variables of less importance, but of some effect in some localities, have been the proportion of nutrients derived from silage; the proportion of feed fed while on pasture during the summer season; the age and weight of the cows; and their quality as indicated by their value per head. The breed of cow was considered in several studies, but was found to have only a negligible influence on production after other factors were allowed for. In spite of the fact that no measures have been found satisfactory for the nutrients the cows obtain from pastures, multiple correlations ranging up to 0.90 have been obtained in these studies, indicating how much the average production of a herd is dependent upon the physical conditions and practices.

Similar correlation studies of the influence of physical input upon output have been made in the case of potatoes (9, 10), cotton (11), and other crops. In the study of potatoes, yield was found to vary with the amount of seed used, the quantity of manure and fertilizer applied, and the depth of plowing. The regression for the latter factor was particularly interesting, in that it was concave downward, indicating that maximum yields were secured with a certain depth of plowing, and that plowing either deeper or shallower decreased the yield.

In the study of cotton, the quantities of mixed fertilizer and of nitrate of soda used were considered as separate variables; the quantity of calcium arsenate applied was considered, and also the fertility of the land as indicated by the yield of other crops, notably corn. The results in this problem raise two interesting points which illustrate some of the logical problems which come up in correlation analyses. The arsenate influences the yield through killing the boll weevil. In the year studied there was a heavy weevil damage on untreated fields, and the applications of poison increased the yield very materially. But these results indicate nothing of how much influence poison would have on yield in years when weevil damage was lighter. It would be necessary to repeat the study over several years with varying weevil damage, and then relate the differences in the effectiveness of poison to differences in the climatic factors which affect the weevil infestation, before it would be possible to judge in any particular year whether or not it would pay to use poison that year—and the prices both of poison and of cotton would enter into the final consideration.

The inclusion of yields of other crops as a factor in the multiple correlation raises another interesting logical point. The net regressions show that, with other factors remaining the same, farms with high yields of other crops also tend to have high yields of cotton. This might be interpreted as indicating that farms with high yields of other crops also have high native fertility, and that in eliminating this factor the results as to the effect of using the other factors have been made more dependable. But it may be that the high yields of other crops are partly due to high fertilization, either during the same year, or in previous years. In eliminating the increased cotton yield associated with high yields of other crops, then, we might really be eliminating part of the result of high fertilization of cotton. The simultaneous determination of the relations by the method of multiple correlation tends to allow for these inter-relations, if they exist, but whether it does so completely in any given case may still be queried. In the particular case cited, collection of additional information as to fertilizer applied on each field in previous years would give a more positive answer to the question as to how much was native fertility and how much was the result of previous treatment, and so give a real solution to the logical dilemma.

Weather Conditions and Crop Yields.—Another type of complex physical relationships which has been satisfactorily treated by multiple correlation is the relation of weather factors to crop yields. The yield problem of Chapter 14, taken from the work of Misner (12), and the potato yield problem of Chapter 20, from the work of Waugh (13), have already been discussed at length. Other problems of the same sort were early studies of the relation of rainfall in July and August to the size of the Illinois corn crop (14); studies of the influence of rainfall and temperature during the growing season on cotton yields (15); studies of the influence of precipitation, temperature, and relative humidity on spring wheat yields (16); and many others which might be mentioned. One interesting study related the weather during the winter to the yield of cotton in the South. This study showed that extreme cold tended to exterminate the boll weevil and so increase the yield of cotton (17). In spite of the fact that the correlation was practically perfect during the brief period of six years for which the study was made, the author did not believe that he had explained all of the causes of variations in the yield of cotton, and modestly refrained from concluding that he had a perfect forecaster of cotton yields. This was fortunate, as after giving an excellent forecast of yield for one year, cotton yields the second year were diametrically opposite to the expected yields—with a departure of many times the standard error of the previous years. This case is interesting as indicating the limited meaning of computed standard errors in time series, and as further indicating that a result which is not sensible logically cannot be trusted as the sole basis for forecasting, no matter how high the correlation. As it was this same investigator who had previously worked out the influence of weather conditions during the growing season on the yield of cotton in individual states, he was forewarned as to the significance of his apparently perfect forecaster, and he was duly cautious in interpreting its meaning. The result was certainly important, however, as indicating that weather factors prior to planting time may be related to subsequent out-turn.

A somewhat different approach to the crop yield problem has been taken by R. A. Fisher (18). In studying wheat yields at Rothamsted, he pointed out that it really made little difference to the growth of the crop whether a given rain occurred on April 30 or May 1, yet if the rainfall were studied by monthly totals the assumed effect might be quite different. Furthermore, if weekly periods were considered for all the different factors, the number of different constants in the regression equation might readily exceed the number of observations. He therefore devised a method of determining the differential relation of rainfall and yield, so as to determine the rate of change in yield with the rate of change in rainfall at any season of the year. The differential equation required a sufficiently small number of constants so that it could be accurately determined from the observations at hand. The resulting smooth curve for the change in yield with changes in rainfall showed that the maximum effect was in fall and in spring, with less effect during the winter. With rainfall through the year the only weather element considered, correlations ranging from 0.32 to 0.63 were obtained for various test plots. Although this method does not take into account joint effects of climate at different seasons (as did the potato yield problem used in Chapter 21), and the method of analysis is more complicated mathematically than any of those presented in this text, the suggestion of determining differential regression equations is a new one, and may in the future open up new possibilities of accurate and complete analysis.

Relation of Physical Characteristics of Samples to Chemical Characteristics.—A quite different application of correlation analysis has been in determining the extent to which the chemical properties of a given sample were related to, or could be estimated from, observable physical properties. The estimation of the protein content of wheat from the proportion of vitreous kernels, used as an illustration in Chapter 6, is taken from a much more comprehensive series of studies (19) in which the weight, the percentage of vitreous kernels, and the region of the country from which the wheat came were all found to have significant influences. In addition, it was found that the relations changed slightly from year to year, so that further work remains to be done to determine the influence of differences in climatic factors on the relation between physical and chemical properties.

A somewhat different study, but also within the same general field as the last one mentioned, related the volume of bread a given quantity of flour would produce, to the gluten content of the wheat and of the flour (20). Correlation was also used to determine the extent to which the digestible composition of different cuts of meat could be judged from the visible proportion of fat (21). Studies such as these illustrate how statistical methods may be used to generalize from the results of many tests, even where the tests themselves were carried out under the carefully controlled conditions of exact scientific experiment.

A somewhat different application of statistical methods to the interpretation of data secured from exact scientific measurements is in the astronomical problem of the relation of the brightness, intensity, and distance of the stars. Careful investigations in this field (22) have leaned heavily upon correlation analysis for their final conclusions.

These last two types of problems deal with purely physical relations, which remain the same, or at worst change only gradually, over a series of years. The idea of a statistical universe which is being sampled may therefore have some application, though it is sometimes a limited one. But in the next type of problem, though the universe is stable at any one time, it may change radically from year to year, so that conclusions for one year may not be at all applicable to those of succeeding years.

Relation of Farm Organization to Farm Income.—The question of what organization will produce the best returns for the farms in a given locality is one that has been given extensive statistical investigation, by correlation means and otherwise. In particular, studies of farm income in Pennsylvania (23); Iowa (24); Virginia (25), to mention specific cases, have made extensive use of multiple correlation analysis. Such factors have been considered as the size of the farm, the acreage in each of the principal crops, the size of the important livestock enterprises, the efficiency of crop production and livestock production, and the capital invested. In general it has been found that about half the variation in earnings from farm to farm in the same year can be explained by such objective measures of their organization and management as just mentioned. The multiple correlations with income range up to a maximum of about 0.75 to 0.80. In addition, it has been found that the size of the dominant enterprise and the efficiency with which it is conducted are usually the most important factors affecting returns. Thus on Iowa hog farms (24) the yield of corn, the number of brood sows, and the efficiency of hog production are dominant factors; on Virginia tobacco farms (25), the acreage in tobacco, the yield of tobacco per acre, and the quality of tobacco; and on Pennsylvania dairy farms (23), the number of dairy cows, and the efficiency of the dairy enterprise.

Beyond these broad generalizations, however, the results of detailed statistical studies of this type are distinctly limited. In the first place, the results hold true only for the particular year in which the records were collected. Differences in yields from one year to another, changes in the prices of each product and of each cost factor, all modify both the physical and the economic situation, so that many allowances must be made before the results can be applied in another year. Even if satisfactory adjustments can be made, there is still another limitation. Each individual farm is a different entity, and the organization which produces the best results *on the average* will not necessarily be the best for any one individual farm. If it were possible to observe one farm under 100 different types of organization and operation, and record the resulting profit secured under each one, it would then be possible to judge from the analysis of those records what type of organization would yield the maximum returns for that farm under the same price conditions. But with the records of different farms representing, not like entities but entities more or less unlike, the conclusions are not so applicable. Only on the assumption that the observations are drawn from a homogenous universe of similar conditions can the results of statistical studies of farm organization be interpreted to give the best organization for any one farm—and the areas where this assumption is justified are probably very few.

Relation of Economic Conditions to Market Price for a Commodity.—All of the problems discussed to this point have been such that a certain universe might be specified, even though that universe would be likely to change more or less with the passage of time. The problem of prices, though, is of an entirely different character, for there only a single observation can be drawn for any given length of period, and the next period is essentially in a different universe. Even so, however, there is enough continuity to the way that individual persons react in the aggregate, and enough similarity between successive years, so that fairly stable results can sometimes be secured, and where the change in reaction is continuous and progressive, that change itself be made one variable in the analysis.

Annual prices.—The simplest price studies are those which relate the market price for a commodity to the supply for a marketing year. The early work on this line by Moore (26) indicated the general relation of supply to price for corn, hay, oats, potatoes, and cotton. The influence of changing conditions were eliminated mainly by the use of first differences, so the resulting curves were not susceptible of logical economic interpretation. More recent work on potatoes (27, 28, 29), oats (30), and cotton (31, 32) has recognized the influence of price levels, trends in demand, carryover from previous years and the prices of competing products, as factors influencing price along with supply; and multiple correlation or alternative techniques are used to take into account the influence of the different variables. With relatively short periods on which to base the analyses, exceedingly high correlations have been secured in many cases—frequently above 0.95, even after adjusting for the number of constants. When forecasts have been made ahead, however, they have met with variable success, the forecasts in some years working out practically as well as in the period on which the analysis was based, and in other cases missing by wide margins, sometimes by many times the standard error of estimate.

These extreme errors in forecasting seem to be due to the element of fortuitousness in economic events. Thus a factor which has been fairly constant for a number of years, and hence has shown little influence on price, may suddenly become very important—and upset a forecast based on the years in which it was unimportant. In addition to such universally-upsetting changes as the outbreak of the World War, other illustrations of such sporadic and unforecasted events are the sudden decrease in the foreign demand for American hog products in the spring and summer of 1927, or the increasing competition of Indian cotton with American cotton in 1928 and 1929 —events which wrecked promising methods of price forecasting.

Monthly prices.—When monthly prices are considered, more elaborate statistical studies have been possible, with a larger number of individual observations. Although questions may be raised as to how closely the prices in successive months of a staple commodity are really independent of each other, there is no question but that conditions are constantly changing, so that there are some elements of independence between successive observations. Of the studies of monthly prices published to date, a study of cotton prices by Smith (33) is of particular economic interest for its separation of the influence of actual and of prospective supply on price, and of the shifting of the regression curves for these factors through the season—determined as joint functions of the month and of the variable. A study of hog prices (34) developed both an empirical forecaster of prices, and an economic interpretation of the influence of market receipts, storage stocks, competing products, and business conditions, on prices. The correlations were relatively low, however, and subsequent analyses have materially modified many of the conclusions. The monthly forecasts of hog prices based on this study were not as accurate as were forecasts which took a broader range of elements into consideration (35). Recent studies of monthly hog prices in Germany by Hanau (36) along the same line have yielded reasonable results and given forecasts which to date have worked well in practice. A study of monthly prices of dressed lamb (37) which gave a correlation of 0.98 for the seventeen years studied is noteworthy in that the same formula served to estimate monthly prices (from current supply data) for three years afterwards, with almost the same accuracy as during the period studied. This analysis considered monthly per capita supplies, price level, competing products, and business activity as independent factors, and determined trend and seasonal variation while simultaneously eliminating the influence of other factors.

The studies mentioned include only a small portion of the statistical studies of price which have been made in recent years, but indicate some of the many ways in which statistical analysis has been used in this field—and also some of the dangers and pitfalls that beset the investigator. Price analysis is the last place to apply statistical methods without thorough logical and economic analysis of the particular problem.

Weekly or daily prices.—For very perishable products, where supplies and prices may fluctuate widely even from day to day, price studies may deal with the average prices for a week or even for an individual day. Representative statistical studies of this type are those of watermelons by Hedden and Cherniack (38), and of peaches by Kantor (39). In both of these studies it was necessary to take into account a regular variation in demand from day to day of the week. Other factors influencing demand were also considered, and it was found that temperature had a marked influence on the price that would be paid for a given supply of watermelons. These short-period studies both related to an individual large market—New York City. Similar studies have been made for other markets and other products.

Relation of Characteristics of Different Lots of a Commodity to Prices at Which They Sell.—All of the price studies which have just been discussed studied the reasons for the change in prices from time to time, for lots of the commodity of uniform or of average quality, and at the same stage of the marketing process. As has been pointed out, only one observation can be drawn from each successive universe. A type of study which presents different statistical problems is that of determining why different lots of the same commodity, sold within a given period and at the same stage of the marketing process, should sell for different prices. In this case there is a true universe—all the sales of the specified kind taking place within the specified period—and as large a sample as is desired can be secured, up to the limits of the universe. The studies of land prices previously mentioned are one example of this type of analysis; and the study of the relation of the price of apples to size, insect injury, and scab, used as an illustration in Chapter 22, is another example.

One of the most interesting studies of this type related the prices of different lots of asparagus to the length of green color in the stalk, the number of stalks in the bunch, and the uniformity of the stalks (40). The results of this study, presented very effectively in pictorial style (as reproduced in Fig. 66), have had a marked influence on the practices by the producers who supply the Boston market, and have led to further experimental investigation as to how to produce asparagus with the desirable qualities (41). Similar studies have been made of the influence of size, color, interior quality, and type of carton on the prices received for eggs sold at retail, for both the New York and Philadelphia markets (42), and for the Wilmington market (43).

One logical point which cannot be overlooked in studies of the effect of quality on price, however, is that the premiums paid for

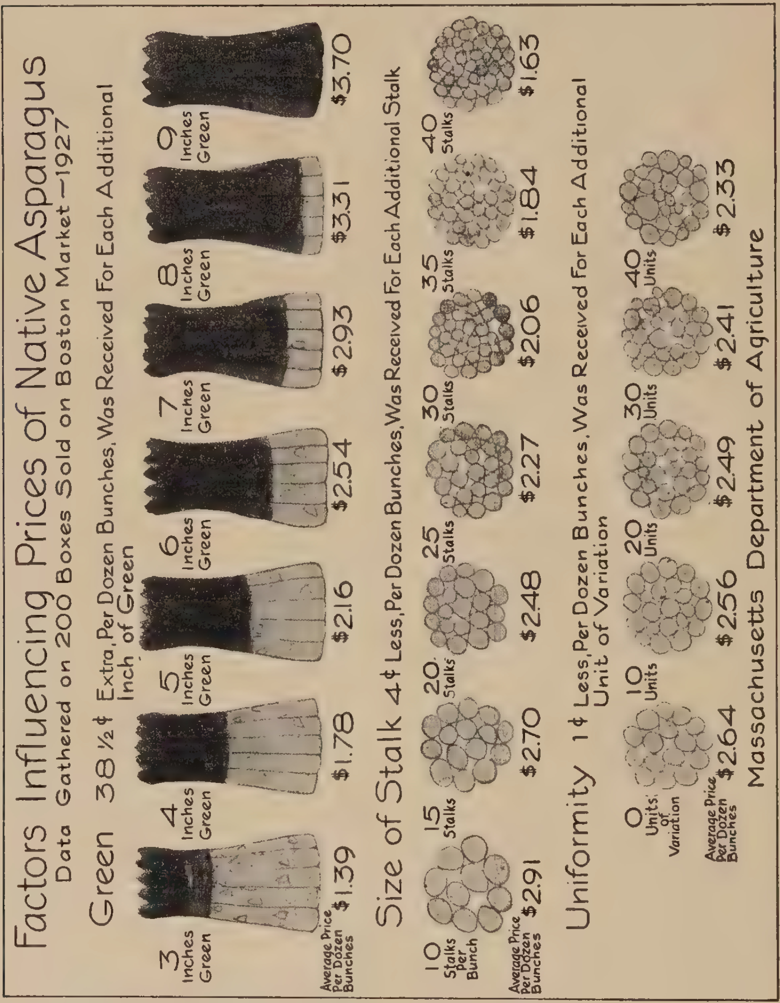

FIG. 66.—A pictorial presentation of conclusions reached by a multiple correlation study. (From Frederick V. Waugh.)

high-quality lots may vary from time to time with differences in the relative supply of products of the different qualities. That is to say, though the conclusions as to the effect of quality upon price do apply in the universe from which the observations were drawn—with certain conditions as to the supply of the different sizes and qualities—

they may not apply in a different universe in which the circumstances have changed. Other studies have therefore attempted to determine not only how the prices vary for different qualities under the set of supply conditions at one particular time, but also how the premium or discounts varied from time to time with differences in the supplies of each quality. Thus studies of the influence of protein content, weight per bushel, dockage, and grade on the prices received for different cars of wheat (44) have shown that in crop years when high-protein wheat is very scarce, a wheat of high protein commands a marked premium, and that factor is much more important than weight; whereas, in years when high-protein wheat is more plentiful but much of the wheat is under weight, the weight factor becomes relatively more important, with the protein premium becoming of much less significance. With records of more than a thousand cars per year for several years, the changes in premiums were determined from month to month, using two joint functions, one for month and protein content, and the other for month and weight per bushel (44).

The effect of varying supplies of different sizes and varieties has also been studied in the case of peach prices (39). Here it was found that the premium for competing varieties changed with the supply of each, a variety which sold at a premium when only a small portion of the total supply, selling at a discount when it exceeded a certain percentage of the supply. The premium for peaches of large size, however, tended to persist in spite of increased supplies, though it was reduced somewhat when the proportion of large size peaches increased.

These last two groups of studies illustrate the way in which changing universes (in time series) may yet be brought within the purview of statistical analysis, and conclusions may be reached which will be of value in new sets of circumstances. If the complex of conditions changes from time to time because of factors such as differences in supply of different sizes, qualities, or varieties, or recurring differences in demand from day to day through the week, or from month to month through the year, which factors can be objectively taken account of and their influence measured with respect to the dependent factor or with respect to the influence of other variables on the dependent factor (joint relationships), then the fact that the circumstances are changing ceases to be a "bug-a-boo," because the reasons for the changes may be determined and allowed for. Just how far the conclusions from such analyses will hold under changed conditions depends upon how adequately the real causes of the changes from time to time have been determined, and how much unaccountable dynamic or evolutionary change there has been and may be. But even so, this approach seems the hopeful one in treating the baffling problem of changing conditions in time series; and it may yet be possible to apply laws of sampling and to make statistical forecasts for these cases with the same confidence that they can be made for stable universes.

Other Price Studies.—Other types of price analysis studies which may be mentioned briefly are those of differences in prices between different points in space or of different points in the marketing process. Correlation analysis has been applied to the first of these problems (45) in studying the relative influence of changes in freight rates, location of supplies, and price level on the margin between potato prices in Minneapolis and New York. Some studies of marketing costs (46, 47) have indicated the influence of size of creamery, distance of haul, and methods of operation on creamery costs and hence on prices received by farmers for their cream; but the general subject of the relation of prices of the same product at different points in the marketing process has not otherwise been investigated, except in the most general way (48).

Another variety of price study is in determining the influence of prices on the quantity of a product moved into consumption. In making studies of this sort for milk (49), it has been found that season of the year, day of the week, holidays, changing food habits, and income of the consumer, all have more influence on consumption than do price changes; but after these are eliminated a slight but significant change in consumption with change in price may be found. With cotton, on the contrary (32), price alone, with an upward trend in demand, almost completely determines the quantity consumed throughout the world; while the quantity consumed in the United States is also influenced by the general level of industrial activity (50). A similar relation for consumption of hog products to price was shown as an illustration in Chapter 6, Table 27. A parallel type of study indicates the effect of price on the quantity of cotton carried over at the end of the season or withheld or used by the producers. Thus it has been found that in years of low potato prices producers feed or waste much larger quantities, whereas when the prices fall below certain points much of the supply is left in the ground undug (51).

In all of these price studies it must be recognized that logically price does not of and by itself determine consumption, carryover, and waste, nor does supply alone determine price. Instead there is a continuous dynamic balance of all the factors, which has been aptly described by the great economist Alfred Marshall as the closing of a pair of shears, where neither blade alone does the cutting. When, however, the relations have been analyzed step by step in the various ways which have been described, the different relations may then be pieced together in a harmonious whole which is logically consistent and which gives concrete statement to the economic hypotheses concerned (52).

Relation of Changes in Production to Prices and Other Factors.—Another type of problem in which prices are involved, but only as independent factors, is in studying the influence of price changes on changes in production. The distinctive characteristic of these studies is that the prices in one period must be related to production in some subsequent period or periods, the length of lag depending on the technological length of the production process, and on the time it takes producers to respond to changes in prices. One of the first studies of this type related cotton acreage to prices for the previous season (53). Subsequent experience showed, however, that continued high prices for two seasons might have a different influence than for a single season alone (54). Studies of hogs showed that it took 18 months for differences in prices to be reflected in market receipts (34). The price of corn was found of equal importance with the price of hogs causing changes in hog production. Hog production has also been studied by different type-of-farming areas, and this detailed study has shown marked differences in the responses to prices in different areas, depending on the position the hog enterprise occupied in the farming system. The weather conditions during the farrowing season in the spring, and the relation of corn prices to hog prices during several critical periods in the production process, were found to be important factors (55). The production of milk has likewise been found to respond to changes in the relation of the price of the milk to the costs of feed-stuff (56). There is a short-time effect, due to changes in the intensity of feeding, and a long-time effect, due to changes in the number of cows (57). With potatoes, prices of two years preceding are important, as well as prices for the year before. The responses for potatoes are quite parallel between different areas, though there are some important differences reflecting differences in the position the enterprise occupies in the farming system (54). In the case of some crops, like flax in the spring wheat belt or sweet potatoes in the cotton belt, the prices for the major crop of the region have as much influence on subsequent acreages of these competing crops as do the prices of the crops themselves. In other cases, yields or per-acre returns for preceding years must be considered, as well as prices alone. The general price-level of competitive products or of all commodities has also usually been considered in judging the significance of a particular price.

In most of these studies of production responses, it has been found necessary to state the subsequent acreage or production as a percentage of, or as an absolute increase or decrease from, the acreage or production of the preceding year or production period. Stating the relation in this way recognizes the fact that the farmer or other producer must plan the next year's operations not with reference to any hypothetical normal or average, but with reference to the actual production situation of the current year. In many cases, as is illustrated in Fig. 67, a very high price will not call forth any larger increase in production in the following year than will a moderately high price, owing to the inability of the producer to expand his operations more than a certain extent in any one year. In this respect this type of price study is quite distinct from other types, for in many studies of the response of prices to supplies more satisfactory results have been secured by working with the absolute figures rather than with changes from year to year.

Miscellaneous Agricultural Problems.—Another group of studies has investigated the relation of the physical characteristics of plants or animals to their ability to produce. Studies of dairy cows, by Gowen (58), restricted to simple two-variable correlations, have indicated that most of the factors in the physical conformation of dairy cows have little or no relation to productive ability. Studies of the relation of the size and shape of corn kernels, ears, and plants to weight of the grain (59), and multiple correlations by Richey which took the actual yielding ability as the criterion (60) have led largely to the same result. These studies indicate that many of the time-honored points which have been stressed in agricultural show competitions and in breeding selection have no utilitarian significance, and have led to a new stress on performance records rather than physical appearance as the ultimate test.

Correlation in Psychology and Education.—Correlation and multiple correlation methods have been widely applied in educational and psychological investigations to the study of such problems as the relation of grades in one subject to grades in another (61), or the scores on one mental test to scores on another (62), or the relation

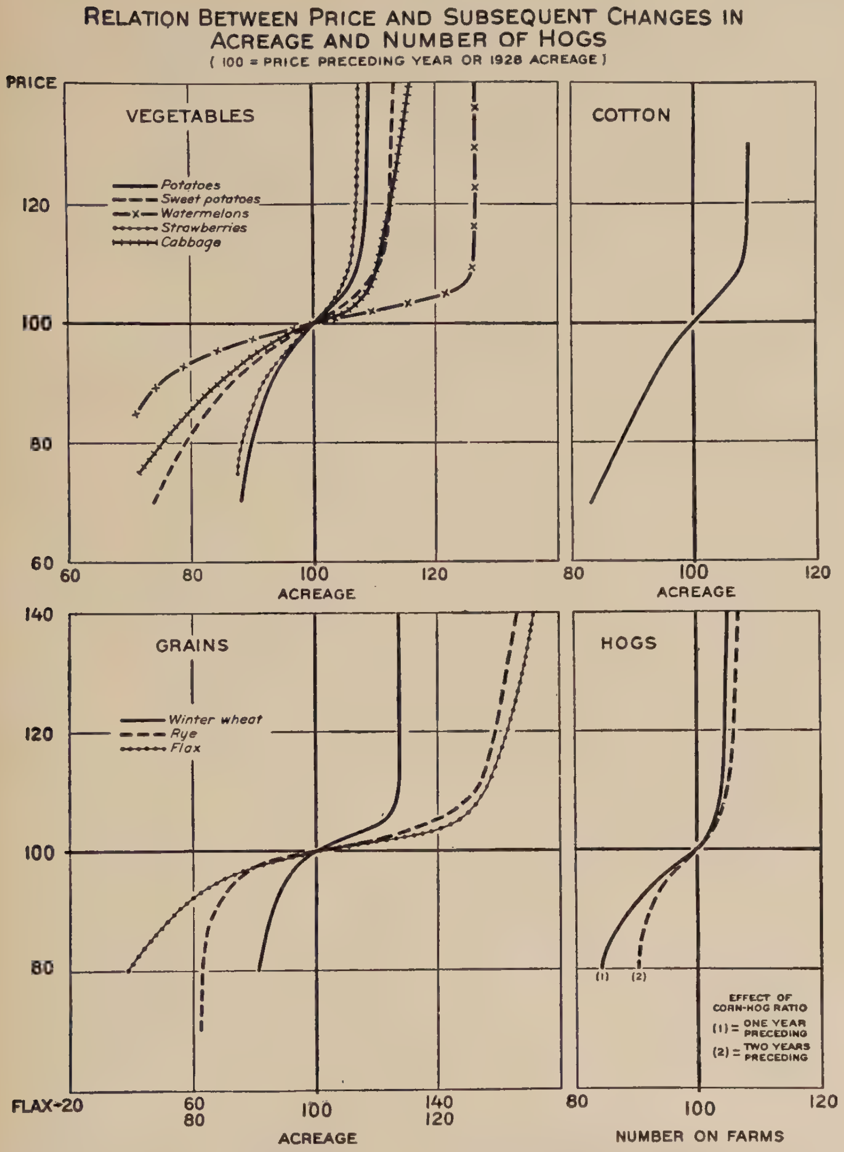

FIG. 67.—Changes in acreage or production with changes in prices received, for different agricultural products. (From reference 54, by Louis H. Bean).

of scores on mental tests to success in the school room (63) or in later life (64). Studies have also been made of the relation of mental and physical characteristics to success in different occupations, such as studying the relation of the relative success of individual farmers to their training, schooling, initiative, business ability, etc. (65). This latter study, which indicated that approximately half of the differences in farmers' financial success could be accounted for on the basis of individual differences in the men, has a tantalizing tie-up with the studies of farm management, which show that roughly half of the differences in income can be explained by the way the farms are run. This would appear to indicate that by considering both the characteristics of the farmer and the way the farm is organized and run, it would be possible to account for *all* of the differences in income. But of course the men with superior mental equipment may tend to be the men whose farms were organized and run in a superior manner, in which case there would be two different measures of the same thing.

In most of the cases in which correlation analysis has been applied to psychological problems, it has been used primarily to measure closeness of relationship, rather than to obtain a basis for estimating one variable from another. In studies of this type even a small correlation may be important, so long as it is large enough so as not to be due to random fluctuations. Thus one study reached the conclusion that even in groups of the same economic and social status, there is a small negative correlation between number of children per family and intelligence (66). The psychologists and the biologists might have a warm argument, though, as to which was cause and which was effect! In another study, in which a given test was repeated, with twice as much time to complete it the second time, the scores made on the two trials were correlated, and correlations of 0.76 to 0.91 were found. These correlations were made the basis for concluding that the tests determine power alone, rather than speed (67). Inasmuch as a correlation of 0.76 means that nearly half of the variance in the two factors is *not* associated, it might be questioned whether this interpretation is altogether satisfactory. Here the use of r ($= 0.76$) instead of d ($= 0.58$) leads to overstressing the significance of the observed correlation. Many other applications of correlation or partial correlation in psychological research (68, 69, 70) illustrate the usual tendency to depend on correlation coefficients, rather than regression equations, as the means of expressing relationship.

Interesting results have also been secured by the application of correlation methods to problems on the border-line between psychology, sociology, and political science. Thus in a study of factors influencing the attitudes of mothers toward sex education, it was found that a number of measures of previous environment showed no significant correlation with the mother's attitude; but that there was a significant correlation between their opinion and the amount of sex education given their children (71). Another interesting study on the political-sociological border-line determined the inter-correlations between the quantities of information, misinformation, and prejudice possessed by college students, and their grades, and their conservative or radical political positions. High prejudice, high misinformation, low grades, and conservatism were found to be associated; and likewise low prejudice, good grades, low misinformation, and radicalism. The correlations were low in all cases, however (72).

The use of correlation methods in the field of education and psychology has been hampered by the fact that in many cases the factors dealt with are not tangible facts which can be objectively measured, but are intangibles which can only be roughly approximated by some process such as ranking. Assuming anything approaching a normal distribution of the factor considered, ranking tends to make the true difference between successive individuals in the series much less in the central portions of the array than in the extreme portions. Furthermore, the ranked series is a discrete series, with the possibility always present that the 6th item in order, for example, may exceed the 7th item by 10 times the amount that the 7th exceeds the 8th, or *vice versa*. Both of these difficulties are apparent in the accompanying set of data.

GRADES RECEIVED BY 24 PERSONS TAKING AN EXAMINATION IN STATISTICS

Rank	Grade	Rank	Grade	Rank	Grade	Rank	Grade
1	99	7	94	13	85	19	81
2	98	8	91	14	85	20	80
3	98	9	90	15	85	21	77
4	98	10	88	16	83	22	75
5	97	11	87	17	83	23	69
6	94	12	85	18	82	24	68

The difficulties enumerated have made psychological workers and educators feel that the standard Pearsonian methods of correlation (those presented in Chapters 4 and 5, and 12 and 13 of this book) are not applicable to their data, and have led to the development of various alternative methods, such as the Spearman "foot-rule correlation" for ranked data (73), and other similar short-cuts. It is not evident that these new measures meet the difficulties enumerated, and furthermore they give measures of correlation which differ from the

Pearsonian coefficients for the same data. The use of curvilinear regressions, as discussed in Chapter 7 and subsequent chapters, partly meets the difficulties in handling such data, since the effect of a varying significance of the unit of measurement in different portions of the range may result in transforming what would otherwise be linear regressions to a non-linear shape. That does not, however, meet the difficulties of the discrete or ''jumpy'' quality of ranked values; nor does it seem that any other statistical treatment is likely to completely do so.

Where the dependent variable is definitely discrete, so that two or more categories can be recognized, but no continuous variation can be assumed, correlation methods are clearly inapplicable. Special statistical measures of association, parallel to the correlation coefficient, have been worked out for such problems (74).

No attempt has been made in this book to treat the special correlation methods developed in educational and psychological work. Instead, it has been restricted to the analysis of dependent variables which were continuously variable, or which could be logically thrown into that form, with the idea that only when changes can be defined and measured in true variable quantities, can the conclusions reach scientific exactness in quantitative statements.

Correlation Analysis in Other Fields.—The types of problems which have been discussed do not begin to exhaust the uses which have been made of correlation, simple or multiple, in research work. Since they are drawn largely from the author's own range of interest, they are heavily weighted by the agricultural or even the agricultural economic field. Random examples of correlation work in other fields are the use of multiple correlation to obtain a definite formula for forecasting pig-iron production (75); to determine the extent to which freight rates are influenced by the factors of terminal charges, length of haul, expense of operation, and other factors (76); or to determine how far meat sales in different branch houses are influenced by local conditions of demand, and what a reasonable quota might be (77).

Many more pages might be filled with the details of studies such as those discussed. But probably enough has been presented to illustrate the wide range of problems in which the use of statistical analysis sheds new light on the relationships present and their significance. It may be hoped that these illustrations have developed the necessity for careful logical analysis of each problem to which statistical analysis is to be applied, and have indicated the need for both good theoretical knowledge of the field in which the problem lies, and thorough technological knowledge of the elements involved in the particular problem. The technological knowledge is particularly important in selecting the different factors or in deciding on their statement or interpretation.

No attempt has been made here to list all of the significant statistical studies in any one of the fields discussed, or to evaluate their importance. Instead, the studies mentioned have been selected solely to illustrate various specific points; in many cases a significant study has not been referred to because the point was already covered, or a relatively unimportant study has been mentioned because of its pertinence to a particular topic. This discussion should therefore not be regarded as a critical evaluation of the work in any of the fields touched upon. That has been left for experts in each field. Instead, the comments are solely intended to develop the variety, complexity, and significance of the problems to which statistical analysis may be applied, and the care and thought which are even more necessary than the statistical computations, if the results are to be of lasting value.

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CHAPTER 23

STEPS IN RESEARCH WORK, AND THE PLACE OF STATISTICAL ANALYSIS

Relation of Statistical Analysis to Research.—Statistical analysis is only a tool to be used by the investigator. The analyst must be a worker in some field, or in several; he cannot use his statistical training except in analyzing problems any more than a carpenter can use his skill without lumber and something to be made. Now that the routine of statistical analysis has been discussed, and the types of problems to which it may be applied have been surveyed, it is pertinent to ask just what are the steps in research work, and just where and how does statistical analysis fit into the picture.

The research worker must have an adequate knowledge of the facts, technical and otherwise, of the field in which he is to work. This knowledge is usually insured by the situation that in most cases the worker is a biologist, an economist, a psychologist, or an agronomist, first, and then a statistician only secondarily or in addition. In those cases where his training has been primarily in mathematics or statistics, however, the worker must thoroughly acquaint himself with the facts of the field involved before he can expect to do significant and substantial work.

Stating the Objective.—Given adequate acquaintance with the field, the first step in a particular research problem is setting up the objective of the project. The objective can best be stated in the form of a direct question, such as "Why does lettuce sell for more on some days than on other days?" The more exact and specific the question can be made, the more clearly is the field of the investigation defined. Thus if we make the question read "leaf lettuce sold at retail in Boston" instead of merely "lettuce," the scope of the study is much more definitely indicated. Stating the objective as a question has the important effect of clarifying the issue, and so insuring that the worker is clear as to what he is really trying to find out. It has the further effect of instantly challenging the attention, and of instinctively calling forth mental answers which aid in the next step of the research.

Any research project which cannot be stated as a definite question has not been clearly defined. Starting out merely "to collect figures on lettuce marketing" would not constitute research. Clear formulation of the question to be answered is an essential prerequisite of good research work.

Developing an Hypothesis.—The second step in the development of the problem is a deductive analysis of the question raised to suggest possible answers to the question. This deductive analysis draws on all the theoretical and practical training and experience the worker has. In addition, he may study previous work along the same lines, ask questions of those concerned in the industry, or make brief reconnaissance studies, to decide on the factors which may be involved and to judge of the probable relationships. This phase of the research should lead to the setting up of a definite hypothesis as to the elements which will be involved and of the ways in which they will be related. Thus in the lettuce problem, the hypothesis might be that the supply of leaf lettuce was the most important factor determining the price, and that the larger the supply, the lower the price; that the supply of Iceberg lettuce also influenced the price of leaf lettuce, large supplies of Iceberg tending to depress the price of leaf lettuce; that weather affected the demand, prices for the same supply being higher in hot weather than in cool; that prices of other vegetables, such as tomatoes and cucumbers, might also influence lettuce prices, either as competitive products tending to depress lettuce prices when their prices were low, or as complementary products tending to raise lettuce prices when their prices were low. It might further be supposed that variations in the purchasing power of consumers would affect the demand, and that changes in the general price level of foodstuffs would also have some effect. Finally, it might be supposed that the demand would vary regularly from day to day through the week, due to the purchasing habits of consumers, and from time to time through the year.

The process of developing the hypothesis may be aided by breaking up the main question to be answered into a number of subquestions, each one of which may be further broken up. Thus the initial lettuce question may be broken up into questions such as "Do (the specified prices) vary because of supply? Because of demand? Supplies of what? Leaf lettuce? Iceberg lettuce? Competing products? What are competing products? What makes demand? Weather? Purchasing power? Seasonal factors?" and so on until complete details have been thought out for every phase.

In setting up the hypothesis the investigator should also attempt to think through the probable nature of the relationships. Thus, should it be assumed that the influence of supply of leaf lettuce on price will be constant, independent of other factors, or is the relation likely to change from time to time through the year, or from day to day with the weather?

Measuring the Factors.—Once the hypothesis has been set up, and the various factors enumerated in it have been considered with much care to make sure that every important element has been included, the next step is to secure measurements of the various factors to be studied. This will involve deciding whether the data are to be taken from published records or other secondary sources, or whether they are to be secured first-hand. If first-hand collection is decided upon, further detailed study is involved as to where the ultimate facts are, who has knowledge of them or records of them, and how they are to be collected—by measurement, by direct observation, by enumerators, by schedules, by mail questionnaires, etc. Extended discussion as to the advantages and disadvantages of each method, and the problems involved in laying out a record form, defining the units, securing the records, and checking or editing the reports, are available in standard statistical texts[1] and will not be repeated here. Precautions also need to be observed if secondary sources are used; these precautions also are well discussed in the references just given. Only one point will be developed here, and that is the special need of *completeness* in the records, particularly if original data are to be secured. Once an enumeration or observation has been made, additional data can be secured only at much extra trouble and expense, or in many cases cannot be secured at all. That is one reason why the hypothesis must be carefully studied beforehand to make sure all relevant factors are included, and why the preliminary study and investigation are so important. Factors which are stumbled upon or which suggest themselves in the later analysis may be of value in subsequent studies of the same type but if the essential data are lacking the suggestions are too late to be of any value in the current study.

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[1] Bowley, Arthur L. Elements of statistics, Chapters III and VIII, pp. 18–57, 178–195. 4th ed. 1920.

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In obtaining the basic data it is necessary to decide on the particular items to be measured to represent the hypothetical factors. Are the weather elements to be rainfall, or wind, or temperature? If temperature, average, or maximum, or minimum? If average, what kind of average? And so on through a lengthy number of details, each one of which must be carefully considered in view of the hypothetical significance of the factor, the probable relations involved, and the effect which is expected to be shown.

Studying the Apparent Relations.—After numerical values are available for all the elements, the next step is to make a thorough study of the apparent relationships before proceeding to more elaborate analyses. Both the relation of the independent factors to each other, and the relation to the dependent, must be studied, for, as has been pointed out before, the relation of the dependent factor to an independent factor that is not related to the others can be determined by simple correlation, whereas otherwise multiple correlation might be necessary. (This does not hold, however, if joint functions are present.) It is at this point that the investigator begins to test out the various elements in the hypothetical picture, and to compare the hypothesis with the observed facts. Some elements which were thought to be of importance may prove unrelated, and other variables which were thought of doubtful significance may show important relations. This preliminary examination may even prove the entire hypothesis to be wrong, and necessitate a re-examination of the basic ideas and a re-formulation of the proposed explanation more in line with the facts as observed.

Running a Correlation Analysis

The preliminary examination of the data will provide the basis for setting up the final multiple correlation analysis, if the interrelations are such that such an analysis is finally found to be needed. As included in this analysis, each variable will have a definite place in the hypothesis, and some specific kind of relation will be expected to be found when the analysis is completed. Looked at in this way, the correlation analysis is not the whole of the research project, but is merely that portion of it in which the adequacy of the theoretical hypothesis is tested, and in which the exact relations, as expected in the hypothesis, are measured and determined.

Units in which Variables Are Stated.—Once the variables to be employed in the final statistical analysis are selected, the next problem is to decide in what units to state them. In studying land values, for example, the value of a given farm may be stated as total value, as value per acre of all land, or as value per acre of improved land. Which one to select depends on what other variables are included and how they are to be stated. The total value of the farm might be correlated with the value of the dwelling, the value of other buildings, the acres in cultivated land, the acres in pasture, etc. This would tend to show the contribution *per acre* of each of the acreage elements, and should give a high correlation, since under normal conditions the value of the farm may be expected to approximate the value of the buildings *plus* that of the several tracts of land. In this case the simple or additive regression equation would be quite appropriate, for it would give

Farm value = value of dwelling + value of other buildings
+ (value per acre of cultivated land) (number acres of cultivated land)
+ (value per acre of pasture land) (number acres pasture land)
+ (value per acre of woodland) (number acres woodland) + etc.

But if it were desired to measure the influence of type of road, fertility of land, and distance from town on land value, they could not be so readily included in the same additive equation. For example, if a 40-acre farm yielding 40 bushels of corn to the acre was worth on the average $1,000 more than a farm of the same size yielding 30 bushels of corn per acre, it would not be reasonable to expect that a 160-acre farm yielding 40 bushels of corn to the acre would be worth only $1,000 more than a 160-acre farm yielding 30 bushels per acre, yet that would be the assumption that would be made if yield of corn were added as a factor to the above equation—for that would assume that a given increase in fertility would add the same amount to the value of the farm, no matter how large or how small the farm was.

If the value were stated as value per acre, that would partly solve the difficulty, for a given change in fertility, distance from town, or type of road, would then be assumed to have the same influence upon value per acre no matter how large or how small the farm was. But that would introduce difficulty with other variables. The dwelling, for example, would not become larger in direct proportion with the size of the farm. Very large farms with good dwellings would have a very low "value-of-dwellings-per-acre," and small farms with poor dwellings would also have a low "value-of-dwellings-per-acre." Only some method of determining the effect of value of dwellings on land values separately for farms of different sizes would take care of this difficulty, as otherwise the same measurements would be used for dwelling values which might be different in their effect on land values, with consequent confusion of the results.[2]

The case mentioned also illustrates the need of something other than a pure arithmetic regression equation to express certain cases. If it be assumed that the more fertile the farm, the greater the effect of nearness to town would be, and that the nearer to town, the greater the effect of an increase in fertility would be, that could not be adequately expressed by the regression equation

$$\text{Value per acre} = f(\text{distance}) + f(\text{fertility}) + \text{etc.}$$

The multiplying effects of the two variables upon the value could be allowed for by using the equation

$$\text{Value per acre} = [f_1(\text{distance})]\,[f_2(\text{fertility})]\,[f\,(\text{etc.})]$$

which, for the actual process of computation, can be stated

$$\begin{aligned}&\text{logarithm (value per acre)}\\ &\qquad = \Phi_1\,(\text{log distance}) + \Phi_2\,(\text{log fertility}) + \Phi_3\,(\text{log etc.})\end{aligned}$$

This logarithmic equation, which puts the relations on a *relative* or *proportional* rather than an *absolute* or *arithmetic* base, is a very flexible one and one that can be used in a great many types of problems.

Finally, if the effect of fertility upon land value be found to vary with fertility, say, and the effect of building value with size of farm, not even the logarithmic equation would be applicable. Instead, an equation of the joint function type (note Chapter 20) might be used, such as

$$\begin{aligned}\text{Log (value per acre)} = {} & f(\text{distance, fertility, roads})\\ & + f(\text{value dwelling, size of farm, value barns})\\ & + \text{etc.}\end{aligned}$$

One further consideration is the danger of false results or spurious correlation if the variables are improperly stated. Thus if an attempt were made to correlate the value of farms with three factors (a) the percentage of land in corn, (b) the percentage of land in wheat, and (c) the percentage of land in all other uses, it would be impossible to solve the problem, or else it would give a spurious result. That is because the three factors (a), (b), and (c) would add to exactly 100 per cent in each individual case, and after variation in (a) and (b) had been held constant by statistical means, there would not be any room left for variation in (c). Even if as the result of rounding off the variables there were slight deviations from the 100 per cent total, the results would have little significance, as the practically perfect intercorrelation between the three independent factors would make the measures of their net influence, both regression coefficients and net coefficients of correlation, exceedingly subject to error. Only by dropping out one of the factors, say (c), would significant results be secured. The regressions on (a) and (b) would then also show the effect of (c); for example, the increase in value for each unit increase in (a) would mean the increase due to substituting one unit of (a) for one unit of (c); changing the sign would give the effect of substituting one unit of (c) for one of (a). The same principle would then apply as between (b) and (c); while the increase in the dependent variable for substituting one unit of (b) for one of (a) would be the difference between the two net regression coefficients.

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[2] See the appendix, pages 39–54, of U. S. Dept. Agr. Bul. 1400, Factors affecting farmers' earnings in southwest Pennsylvania, for an example of statistical treatment of a problem of this type.

Steps in Carrying Through the Computations.—After the variables and the form of the equation for the statistical analysis have been decided upon, the next step is actually carrying through the computation. This involves "coding" the numerical values of the variables, that is, reducing them to simpler terms for ease of handling; calculating the extensions; setting up and solving the normal equations; and calculating the standard error of estimate, the coefficient of multiple correlation, and possibly the coefficients of separate determination or of part correlation. Then if curvilinear regressions are desired, the residuals from the linear regression equation will be computed, and the net regression curves determined by successive approximation (or by the graphic short-cut if the conditions are favorable). After the final curves are determined, the standard error of estimate for the curvilinear regressions and the index of multiple correlation are computed. If joint functions are suspected, the residuals are grouped with respect to two or more variables, or studied with respect to compound variables of the Court type, until by successive approximations the final shape of all the functions, simple or joint, has been determined, and the new standard error of estimate and index of correlation computed. As a final step, the standard error of each of the regression coefficients, or of each portion of each regression curve, should be computed and indicated on the regression charts, to indicate the significance to be attached to the results. The standard error of the correlation coefficient or other constants likewise should be determined.

Meaning of Correlation Results.—It must be noted, however, that a statistical determination of the nature of any relation, no matter how complicated the methods used in making the determination or how flexible the type of function allowed for, tells nothing of the *reason* for the relation observed.

Thus the variation in potato yields with differences in early and late rainfall, as determined in Chapter 21, may be due to a large variety of different causes. The plant requires certain conditions as to soil moisture, nutrients, sunshine, maximum and minimum temperature, and relative humidity to make the best growth, and the factors used reflect certain of them. Further, it may be that one set of conditions is required during the first part of the growing period while the plant is developing its leaves and top, and another set later on while it is developing the tubers; and that the rainfall factors used relate in this way to the growth periods of the plant.

There are other possibilities, however. The yield of a plant is affected not only by the weather conditions as they directly affect the development of the plant itself, but also as they affect the development of insects and diseases that prey on the plant. For example, the peculiar relation of potato yields to early and late rainfall considered jointly, as shown in Figure 65, might reflect the relation of late rainfall to potato diseases. With 16 inches of early rainfall, and 3 inches of late, a yield of 240 bushels would be expected; with the same early rainfall, as the late rainfall increases, the probable yield falls until with 6 inches of late rainfall it is under 180. The heavy early rainfall may stimulate good growth of the top; then if heavy late rainfall should follow it might result in conditions favorable to the development of potato blight, and so reduce an otherwise promising yield.

It is evident that a considerable range of specific technical information is necessary to interpret correctly the results of a correlation analysis, and to develop the reasons for the particular relations which have been found to exist. For best results this technical knowledge must be drawn upon in the early stages of the investigation, to aid in selecting and stating the variables to be considered in such a way that the functional relations, when found by appropriate statistical means, would adequately represent the technological elements present and so be capable of a logical technical interpretation. The correlation analysis itself can never provide the interpretation as to cause and effect. It can only establish the *facts* of the relations—for the meaning of those facts, the investigator must look elsewhere.

The way in which correlation analysis establishes the facts of relationship and nothing else, may be illustrated by a specific example. If the number of automobiles moving down 16th Street in Washington, D. C., for each 15-minute period through a given 12 hours is correlated with the height of the water in the Potomac River during each of the same periods, a definite correlation will be obtained. On some days this correlation would be so high that its probable error would indicate that it would be very unlikely that it could have occurred by chance. However, if on the basis of this correlation one were to attempt to forecast the flow of traffic from the height of the water, he would find his forecast sadly in error if he happened to select a day on which the street was closed for traffic repairs, or when the water was high because of a flood. This is a case in which it is perfectly obvious that there is no direct causal relation between the two phenomena. Yet there is real correlation between them because they both are influenced, though very remotely, by the same sequence of cosmic events. The rising and the setting of the sun have a very definite influence on the movements of persons and therefore on the flow of traffic, whereas the rising and the setting of the moon likewise have a definite influence on the height of the water. At any particular time there will be a certain specific though complex relation between the rising and setting of the sun and of the moon. (Of course, this relation is changing constantly from day to day. That introduces a further complication that will only be mentioned here.) This illustrates a case in which real and significant correlation reflects causation by a common factor or factors, yet gives no inference as to direct causal connections. Many similar cases are met with in practical work in which the correlation between two variables is due to both being influenced by certain common causes although neither may in any conceivable way influence the other. This illustrates again the need for clear logical thinking and for a technological basis for the interpretation of the statistical results, which can measure the relationships, but of themselves can tell nothing of cause or effect.

Interpretation and Statement of Results of Correlation Analysis

Having completed the statistical analysis of the relations—the extent and complexity of which will depend upon the nature of the problem, the number of observations available, the importance of the relations, and the facilities available with which to work—the next step is to translate the statistical results to intelligible non-technical statement. This may go only so far as simple regression charts or estimating tables of the type shown at the end of Chapter 13, or of carefully worked-out pictorial statements such as shown in Fig. 66. After the results are reduced to intelligible form—intelligible, that is, at least to the investigator—they should be carefully compared with the original hypothesis. If hypothesis and the statistical results do not agree, the hypothesis must be carefully examined to see if it may logically be re-stated so as to be consistent with the facts as found; and the analysis must be carefully studied to see if there are any loopholes in the way the facts are stated, or in the way the problem has been worked through, which may be responsible for the results. (The case cited on page 318, Chapter 22, is an example of mis-statement of the variables.) If the hypothesis and results are found to be consistent, or if without doing violence to either, they can be brought into reasonable agreement, the project may be regarded as completed. If such agreement is not obtained, the results may be announced as actual observations inconsistent with what was expected, and subject to further study or independent checks before being accepted as scientific conclusions.

When the hypothesis and the analysis are found to be in satisfactory agreement, all that remains is to interpret the results to those who will be interested in them and have to use them. At this point many investigators fail to take into account the audience for which they are writing. If they are writing a technical paper for a scientific journal, a full discussion of the methods and techniques used, statistical and otherwise, will be quite in place, so that their fellows may pass on the adequacy of the work. If, instead, they are writing a general or a popular report for an audience which is only interested in what they have discovered and what it means, details of statistical technique may be as out-of-place as computations of stresses and strains would be in a magazine devoted to "The Home Beautiful." The plethora of technical terminology in some supposedly popular reports of statistical investigations have led the readers to suspect that the investigator himself did not understand what his results really meant. Unless the conclusions can be translated back into "the King's English," and stated so simply that practical men dealing with the problem investigated can understand what the results mean, the usefulness of the research may be largely wasted.

Summary.—The place of statistical analysis in scientific research is no different from the place of any other technical aid the investigator may employ. It furnishes a means of specifically measuring the elements that are involved, and of examining the way in which they are related; but it does not of itself furnish an explanation of phenomena. Except insofar as the effort to reduce the variables to specific numerical statement, definitely related, forces the investigator to think more clearly and definitely about his problem, statistical analysis is not a substitute for logical analysis, clear-cut thinking, and full knowledge of a problem. The methods of analyzing complicated relations set forth in this book furnish the student a keen tool for investigating complex relationships; but like all keen tools, they may yield unsatisfactory or misleading results if employed carelessly or heedlessly. Statistical analysis is not a substitute for careful thinking and skilled workmanship in research work; instead, it is an aid which may make that thought and skill even more productive of worth-while results.

APPENDIX 1

METHODS OF COMPUTATION

Coefficients of Correlation and Regression.—Many of the operations described in the text may be performed much more rapidly by short-cut methods. One such method, for computing σ_x has already been given. (Note 1, Chapter 1.) When the value Σx^2, required in the normal equations, is desired instead, that may be calculated from a frequency table by the same method, by use of the relation

$$\Sigma x^2 = \Sigma(d^2F) - n\left[\frac{\Sigma(dF)}{n}\right]^2$$

A similar short-cut may be used in computing the product-sums, Σxy, required to determine coefficients of regression or correlation. The first step is to construct a double-frequency table. Such a table is known as a correlation table, since it shows the nature of the relation between the two variables in much the same way that a correlation-chart or dot-chart does. The following correlation table, Table 85, is prepared from the hay-stack data used as an illustration in Chapter 20.

After the number of items falling in each subgroup have been determined, the number of entries in each row are summed, giving the total frequencies with respect to X_1. These frequencies, denoted F_1, are shown at the right in the table, in the column headed "All values of X_2." The entries in each column are similarly added, and the totals entered at the foot. These entries give the frequency distribution with respect to X_2, and are, therefore, denoted F_2 (frequencies of X_2). A central group is then selected for each variable, and the departures of the other groups, above or below that central group, are shown in the "d_1" and "d_2" column and line, respectively. The usual extensions to compute the Σx^2 for each variable are shown in the final columns, d_1F_1 and $d_1{}^2F_1$, and lines d_2F_2 and $d_2{}^2F_2$. As their designations indicate, the entries under these heads are obtained by first multiplying the F_1 entry by d_1, giving d_1F_1, and then multiplying that again by d_1 to give $d_1{}^2F_1$. The similar computations for F_2 are shown at the foot of the table.

The new step in the table is the incorporation of the column Σd_2F_1 and of the line, Σd_1F. The entries in the column Σd_2F show for each line the sums of the frequencies of each cell in that line multiplied by the d_2 values for each cell. Thus for the first line, the single entry has a d_2 value of -3, so the entry in Σd_2F is -3. The next line similarly has a single entry in the -2 column. The fourth line, though, has the following frequencies: 1 in the -4 column, 1 in -3, 2 in -2, 8 in -1, 4 in 0, and 1 in 1. The respective products, -4, -3, -4, -8, 0, and 1, add to -18, and this value is, therefore, entered in the Σd_2F column.

The Σd_1F line is similarly computed, showing the sum of the frequencies in each cell for each column, multiplied by the corresponding d_1 values. Thus the first column has 1 entry in the -1 line; the second, 1 in the -4, and 1 in the -1 with the sum -5. The third column has 1 in -3, 1 in -2, 2 in -1, 6 in 0, and 3 in 1.

TABLE 85

Correlation Table—Number of Reports Classified With Respect to Both X_1 and X_2; and Calculation of Extensions and Coefficients

Values of X_1	Values of X_2									All values X_2 F_1	d_1	Σd_2F	$d_1(\Sigma d_2F)$	d_1F_1	$d_1^2F_1$
	0.045–0.064	0.065–0.084	0.085–0.104	0.105–0.124	0.125–0.144	0.145–0.164	0.165–0.184	0.185–0.204	0.205–0.224						
0.020–0.099	......	1	......	......	......	......	......	......	......	1	−4	−3	12	−4	16
0.100–0.179	......	......	1	......	......	......	......	......	......	1	−3	−2	6	−3	9
0.180–0.259	......	......	1	......	1	......	......	......	......	2	−2	−2	4	−4	8
0.260–0.339	1	1	2	8	4	1	......	......	......	17	−1	−18	18	−17	17
0.340–0.419	......	......	6	9	5	3	1	......	......	24	0	−16	0	0	0
0.420–0.499	......	......	3	10	10	8	2	1	......	34	1	−1	−1	34	34
0.500–0.579	......	......	......	1	3	10	4	1	......	19	2	20	40	38	76
0.580–0.659	......	......	......	......	1	5	6	3	......	15	3	26	78	45	135
0.660–0.719	......	......	......	......	......	......	......	3	1	4	4	13	52	16	64
0.720–0.799	......	......	......	......	......	1	......	2	......	3	5	7	35	15	75
All values X_1 F_2	1	2	13	28	24	28	13	10	1	120	Σ	24	244	120	434
d_2	−4	−3	−2	−1	0	1	2	3	4	Σ					
Σd_1F	−1	−5	−4	4	13	47	28	34	4	120					
$d_2(\Sigma d_1F)$	4	15	8	−4	0	47	56	102	16	244					
d_2F_2	−4	−6	−26	−28	0	28	26	30	4	24					
$d_2^2F_2$	16	18	52	28	0	28	52	90	16	300					

Calculation of Corrections to Deviations from Mean

	Σd_1F	Σd_2F	$\Sigma d_1^2F_1$	$\Sigma d_2^2F_2$	Σd_1d_2F
Sums............	120	24	434	300	244
Means............	1.00	0.20			
Corrections........	....	....	−120	−4.8	−24
Corrected values...	....	....	314	295.2	220

$$b_{12} = \frac{\Sigma x_1x_2}{\Sigma x_2^2} = \frac{220}{314} = 0.701$$

$$r_{12} = \frac{\Sigma x_1x_2}{\sqrt{(\Sigma x_2^2)(\Sigma x_1^2)}} = \frac{220}{\sqrt{(314)(295.2)}} = 0.722$$

The products -3, -2, -2, 0, and 3, add to -4; and this is the value for the Σd_1F entry for that column.

After all the Σd_2F entries are made, each is multiplied by the d_1 value for the same line, giving the values entered in the $d_1(\Sigma d_2F)$ column. Similarly, the entries in the Σd_1F line are then multiplied by the corresponding d_2 values, and the products entered in the $d_2(\Sigma d_1F)$ line. Each line at the foot of the table is then summed, and the sums entered at the right of the line; and each column at the right of the table summed, and the sum entered at the foot of the column.

The arithmetic may now be checked by the following identities:

$$\Sigma F_1 = \Sigma F_2 \text{ (= 120, in the illustration)}$$

$$\Sigma d_2F = \Sigma d_2F_2 \text{ (= 24)}$$

$$\Sigma d_1F = \Sigma d_1F_1 \text{ (= 120)}$$

$$\Sigma d_1(\Sigma d_2F) = \Sigma d_2(\Sigma d_1F) \text{(= 244)}$$

As a matter of fact, the lines Σd_1F and $d_2(\Sigma d_1F)$ may be omitted, if this check is not to be made. Where many items are involved, however, this is a rapid and accurate check. Only the extensions $d_2{}^2F_2$ and $d_1{}^2F_1$, are then not checked, and these may be readily verified by recomputing.

The next step is to adjust the values in terms of departures from the assumed means to terms of departures from the true means. To do this in organized fashion, the five sums are entered in order, as shown at the foot of the tabulation. The first two items, Σd_1F and Σd_2F, are each divided by the number of cases (120) to give the average departure (in terms of class intervals) from the assumed mean group, that is, values for M_{d_1} and M_{d_2}.

The correction factors are then computed as follows:

$$\Sigma x_1{}^2 = \Sigma d_1{}^2F_1 - (\Sigma d_1F)(M_{d_1}) = 434 - (120)(1.0) = 314$$

$$\Sigma x_2{}^2 = \Sigma d_2{}^2F_2 - (\Sigma d_2F)(M_{d_2}) = 300 - (24)(0.20) = 295.2$$

$$\Sigma x_1x_2 = \Sigma d_1d_2F - (\Sigma d_1F)(M_{d_2}) = 244 - (120)(0.20) = 220$$

This process is shown in tabular form for each item.

The coefficients of regression and of correlation are then computed by the usual formulas, as shown at the foot of the table.

The coefficient of regression shown in the tabulation, is of course in terms of group units. That is, the value $b_{12} = 0.7$ means that for every change of one group interval in X_2 there is on the average a change of 0.7 group intervals in X_1. Since the group intervals are 0.020 for X_2, and 0.080 for X_1, this does not apply to the actual X_1 and X_2 values. Instead, correction must be made as follows:

Regression of X_1 on X_2 in terms of original units = regression in terms of group units $\left(\dfrac{\text{Group interval of } X_1}{\text{Group interval of } X_2}\right)$

In this case,

$$b_{12} \text{ (for } X_1 \text{ and } X_2) = 0.7006 \frac{0.080}{0.020} = 2.8024$$

Hence for each change of 1 unit in X_2, X_1 changes 2.8 units, on the average.

Before a_{12} can be calculated, it is necessary to have the means of X_1 and X_2. The assumed mean of X_1, at the mid-point of the group 0.340–0.419, would be at 0.380. Since the mean of d_1 is 1.00, the true mean lies exactly one group-interval, or 0.080, higher than this, or at 0.460. Similarly, the assumed mean of X_2, at the mid-point of the group 0.125 −0.144, is at 0.135. Adding to this the mean of d_2, or 0.20, times the group-interval of 0.020, gives 0.139 as the mean of X_1. The value of a_{12} may now be calculated by formula (10)

$$\begin{aligned} a &= M_1 - b_{12}M_2 \\ &= 0.139 - (2.8024)(0.0460) \\ &= -1.150 \end{aligned}$$

From the values shown in the table, the regression equation is, therefore, found to be

$$X_1 = -1.150 + 2.8024\, X_2$$

The uncorrected correlation coefficient, r_{12}, has been found to be 0.722, as shown in the table. In making this computation, however, no allowance has been made for the tendency of grouping to exaggerate the departures of the individual cases from the means, which affects Σx_1^2 and Σx_2^2, but does not affect $\Sigma x_1 x_2$. This may be allowed for by applying Sheppard's correction to Σx_2^2 and Σx_3^2. (See Note 1, Chapter 1.) Since the correction is (corrected σ^2) $= \sigma^2 - \frac{1}{12}$, the correction for $\Sigma x_1^2 = \Sigma x_1^2 - \frac{n}{12}$. Applying this correction to both Σx_1^2 and Σx_2^2, in the formula for r_{12}, the value of r_{12} comes out 0.747, a definitely higher value. In this case only 9 groups have been used for X_2, and only 10 for X_1, so the correction for grouping is important. In most practical work, at least 20 to 30 groups should be used for each variable; and when that is done, the application of the correction for the fineness of grouping becomes of much less significance.

Although the correlation computed with Sheppard's corrections (and adjusted for the number of observations by equation 25) gives the best estimate of the true correlation in the universe from which the sample was drawn, the formulas for the standard error of correlation coefficients are all based on the uncorrected formula. If any test of the significance of the observed correlation, such as Fisher's z-transformation, is to be applied, the uncorrected value should be used.

The regression coefficient, as well as the coefficient of correlation, is changed slightly if Sheppard's correction is applied. Thus, using the correction,

$$b_{12} = \frac{\Sigma x_1 x_2}{\Sigma x_2^2 - \dfrac{n}{12}} = \frac{220}{314 - \dfrac{120}{12}} = 0.724$$

Just as with the correlation coefficient, the larger the number of groups, the less influence the correction has on the calculated values. With 30 or more groups, it is ordinarily neglected.

There are many other ways in which the coefficient of correlation may be calculated. Thus it may be shown (technical note 1) that if

$$X_3 = X_1 - X_2$$

that then the standard deviations are related according to the formula

$$\sigma_3^2 = \sigma_1^2 - 2r_{12}\sigma_1\sigma_2 + \sigma_2^2$$

Hence the correlation between X_1 and X_2 may be found by calculating the difference, X_3, between each pair of values for the two variables, and computing the standard deviation of each of the three series. The correlation is then given directly by the formula

$$r_{12} = \frac{\sigma_1^2 + \sigma_2^2 - \sigma_3^2}{2\sigma_1\sigma_2}$$

Coefficients of Multiple Correlation and Net Regression.—When many variables are to be considered, and a large number of observations are available, the necessary extensions for multiple correlation lines or curves fitted by the least squares method may be made most readily by the use of tabulation cards, one for each observation, with the values for each variable entered on each card. If mechanical or electrical tabulation equipment is available, the values may be designated by punched holes. The cards can then be sorted and tabulated by automatic machines.

The most rapid method of calculating the extensions, where card-tabulating equipment is available, is by a combination of the "digiting" method with cumulative addition. Thus if four variables are being considered, the extensions for ΣX_1^2, ΣX_1X_2, ΣX_1X_3, and ΣX_1X_4, would be secured by sorting on X_1. If each variable were tabulated in two digits (0 to 99) the cards would first be classified in ten groups from 00 to 90 on the tens column of X_1 and the total for each variable computed. The cards would then be re-classified into ten groups from 0 to 9 according to the values in the digit column, and the total for each variable computed. The totals would then be entered as shown in Table 86, starting with the highest value and running down to the smallest.

TABLE 86

Computation of Extensions by "Digiting" and Accumulative Totaling

Value of X_1	Number of items (1)	Accumulations of X_1 (2)	$\Sigma(X_1)$ (3)	Accumulations of $(X_1)^2$ (4)	$\Sigma(X_2)$ (5)	Accumulations of X_1X_2 (6)	$\Sigma(X_3)$ (7)	Accumulations of X_1X_3 (8)	$\Sigma(X_4)$ (9)	Accumulations of X_1X_4 (10)
30–39	2	20	69	690	11	110	1	10	5	50
20–29	0	20	0	690	0	110	0	10	0	50
10–19	20	220	295	3640	32	430	18	190	26	310
0– 9	10	(320)	66	(4300)	14	(570)	16	(350)	23	(540)
−9	3	3	47	47	2	2	5	5	2	2
−8	2	5	28	75	3	5	7	12	3	5
−7	0	5	0	75	0	5	0	12	0	5
−6	1	6	21	96	12	17	2	14	3	8
−5	13	19	183	279	20	37	8	22	14	22
−4	5	24	70	349	6	43	3	25	7	29
−3	2	26	11	360	7	50	1	26	3	32
−2	1	27	7	367	2	52	2	28	2	34
−1	3	30	41	408	2	54	5	33	14	48
−0	2	(32)	22	(430)	3	(57)	2	(35)	6	(54)
Sums	$\Sigma X_1 = 405$		$\Sigma(X_1)^2 = 7076$		$\Sigma X_1X_2 = 915$		$\Sigma X_1X_3 = 387$		$\Sigma X_1X_4 = 595$	

After the number of items and the sums for each group are entered (columns 1, 3, 5, 7, and 9) in the table the columns headed "accumulations" are computed as follows: the first item, times 10, is entered in the top line. Thus the first item in column (1) is 2, so 20 is entered in column 2. The second item in each column is then multiplied by 20, added to the first item in the accumulation column, and the total entered on the second line of the accumulation column. [Since the second item in column 1 is 0, the second item in column 2 is 20 + (10)(0) = 20.] The third item in each odd-numbered column is next multiplied by 10, added to the second item in each adjoining accumulation column, and the total entered on the third line of each accumulation column. [The third item in column 1 is 20, so the third item in column 2 is 20 + (10)(20) = 220.] The same operation is performed for the next line. When this process is completed for all the classifications of X_1 by tens, it is begun afresh for the classifications by digits, without multiplying by 10. The item in the "-9" class, 3, of column 1, is entered in the adjoining accumulation column. The next item, 2, is added to it, and the total, 5, entered in the accumulation column, and so on down the column, entering in column 2 the accumulated total of column 1. The same operation is performed in each of the other accumulation columns, each showing the accumulated total for the column to its left.

In each case the accumulated total for the -0 group is one-tenth that for the $0-9$ group, and is equal to the sum of the particular variable, checking all the computations. That is because each value appears twice: once when the observations are classified according to the first digit of X_1 (in the tens-column), and once when they are sorted according to the last digit of X_1 (in the units column). If there were also a hundreds-column, there would be a third sort for that, and the accumulative totals for the hundreds-groups, with 00 added at each step, would be 100 times as large as for the unit-groups. After the entries in the -0 line have been checked against those in the $0-9$ line, both are enclosed in parentheses. All of the entries in each accumulation column are then added, except those enclosed in parentheses. The totals for each column are then the extensions for ΣX_1, ΣX_1^2, ΣX_1X_2, etc.[1]

By the use of this method, each variable can be carried to 3, 4, or even more digits if desired, yet the extensions be obtained with exactly the same precision as if each individual item were extended separately. The work is very greatly reduced; the extensions, even if 3 digits were used for each variable, requiring at the most only 30 lines, or for 4 digits, 40 lines.

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[1] It is easily seen why this is so. Each item in the 30+ group appears 3 times in the accumulation column, times 10 each time; it, therefore, contributes 30 to the total. Likewise, each item in the -9 column appears 9 times in the accumulative totals from -9 to -1, so contributes 9 to that total. An item of $X_1 = 39$ is represented by 1 in the 30–39 class in column (1), and by 39 in the same class in column (3). It also appears as 1 in the -9 class in column (1), and as 39 in the same class in column (3). Its contribution to ΣX_1 is then $(3)(10) + (1)(9)$, or 39, and to $\Sigma(X_1^2)$ is $(39)(3)(10) + (39)(1)(9) = 1521$, or exactly equal to 39 and $(39)^2$. Similarly, an item of $X_2 = 2$ when $X_1 = 39$, appears in column (5) in the 30–39 line, and in the -9 line. It then appears 3 times in column (6), multiplied by 10 each time, and 9 times in (6), multiplied by 1. Its contribution to $\Sigma(X_1X_2)$ is then $(2)(10)(3) + (2)(9) = 78$, or exactly equal to $(39)(2)$. It is now evident why the entries in the 0 lines are not included in the total—they contribute nothing to the product with X_1.

After the extensions with respect to X_1 had been made as just shown, the cards would be re-classified with respect to X_2, and a similar tabulation and accumulative totals prepared to obtain ΣX_2, ΣX_2^2, $\Sigma X_2 X_3$, and $\Sigma X_2 X_4$; and so on for the other extensions required.

This method is of the greatest value where automatic card-tabulation equipment is available, and a large number of observations are to be treated. Even for hand-operations, however, if the number of observations are very large it can be used to advantage, with the individual items entered on cards or strips for handy classifying and adding.

Use of the Check Sum.—Where a number of different variables are involved, every operation in making the extensions, computing the averages and corrections, and solving the normal equations through to the "back solution," can be verified by an automatic check known as the "check sum." The way in which the check sum is used will be illustrated by a small problem, carried through every step in turn, but it is equally applicable to any other method of tabulation, and is especially valuable with machine tabulation, where it serves as an over-all control on the accuracy of the machine processes.

The check sum as a check in extending.—The values in the following table (Table 87) may be used to illustrate the use of the check sum.

The values in the columns X_2, X_3, X_4, and X_1 are the three independent factors and the dependent factor, which are to be correlated. The values in the column headed "Σx" are the arithmetic totals of the values for the four other variables, and are designated "the check sum."

As the first step, each of these five columns are added. Since, for each line,

$$X_2 + X_3 + X_4 + X_1 = \Sigma x$$

it also holds true that

$$\Sigma X_2 + \Sigma X_3 + \Sigma X_4 + \Sigma X_1 = \Sigma(\Sigma x).$$

Adding the sums of the first four columns together, that gives the same value as the sum of the check sum column, which verifies all the totals.

The first set of extensions is made by multiplying the items in each line by the X_2 item in the first column of that line, giving the values shown under "Extensions with X_2". Since for each line

$$X_2 + X_3 + X_4 + X_1 = \Sigma x,$$

it also follows that

$$X_2^2 + X_2X_3 + X_2X_4 + X_2X_1 = X_2\Sigma x.$$

Then, adding each column, the sums of the four other columns should total to the same value as the sum of the $X_2\Sigma x$ column. Checking up, 1,994 + 25,315 + 14,158 + 14,224 = 55,691, verifying all the calculations.

The other extensions are made in similar fashion, and the sums of each column verified with the sum of the check-sum column, according to the relation

$$\Sigma X_2X_3 + \Sigma X_3^2 + \Sigma X_3X_4 + \Sigma X_3X_1 = \Sigma(X_3\Sigma x),$$

and the corresponding relation for the other extensions. It should be noted that in checking the "extensions with X_3," the value ΣX_2X_3, is taken from the previous set of extensions; in checking the "extensions with X_4," the value for ΣX_2X_4 is taken from the "extensions with X_2," and the value for ΣX_3X_4 from the "extensions with X_4"; and so on for the remaining checks.

TABLE 87

CALCULATION OF EXTENSIONS, USING THE CHECK SUM

Variables					Extensions with X_2				
X_2	X_3	X_4	X_1	Σx	X_2^2	X_2X_3	X_2X_4	X_2X_1	$X_2\Sigma x$
0	136	106	103	345	0	0	0	0	0
1	140	103	108	352	1	140	103	108	352
2	86	108	102	298	4	172	216	204	596
3	115	102	111	331	9	345	306	333	993
4	115	111	95	325	16	460	444	380	1,300
12	161	91	109	373	144	1,932	1,092	1,308	4,476
13	235	109	118	475	169	3,055	1,417	1,534	6,175
14	304	118	123	559	196	4,256	1,652	1,722	7,826
15	224	123	108	470	225	3,360	1,845	1,620	7,050
16	185	108	100	409	256	2,960	1,728	1,600	6,544
17	108	100	88	313	289	1,836	1,700	1,496	5,321
18	193	88	109	408	324	3,474	1,584	1,962	7,344
19	175	109	103	406	361	3,325	2,071	1,957	7,714
134	2177	1376	1377	5064	1994	25,315	14,158	14,224	55,691

Extensions with X_3				Extensions with X_4			Extensions with X_1	
X_3^2	X_3X_4	X_3X_1	$X_3\Sigma x$	X_4^2	X_4X_1	$X_4\Sigma x$	X_1^2	$X_1\Sigma x$
18,496	14,416	14,008	46,920	11,236	10,918	36,570	10,609	35,535
19,600	14,420	15,120	49,280	10,609	11,124	36,256	11,664	38,016
7,396	9,288	8,772	25,628	11,664	11,016	32,184	10,404	30,396
13,225	11,730	12,765	38,065	10,404	11,322	33,762	12,321	36,741
13,225	12,765	10,925	37,375	12,321	10,545	36,075	9,025	30,875
25,921	14,651	17,549	60,053	8,281	9,919	33,943	11,881	40,657
55,225	25,615	27,730	111,625	11,881	12,862	51,775	13,924	56,050
92,416	35,872	37,392	169,936	13,924	14,514	65,962	15,129	68,757
50,176	27,552	24,192	105,280	15,129	13,284	57,810	11,664	50,760
34,225	19,980	18,500	75,665	11,664	10,800	44,172	10,000	40,900
11,664	10,800	9,504	33,804	10,000	8,800	31,300	7,744	27,544
37,249	16,984	21,037	78,744	7,744	9,592	35,904	11,881	44,472
30,625	19,075	18,025	71,050	11,881	11,227	44,254	10,609	41,818
409,443	233,148	235,519	903,425	146,738	145,923	539,967	146,855	542,521

While the check sum would not disclose exactly compensating errors made in different columns, the possibility of such errors is so remote that after the arithmetic has been checked by the comparisons indicated, it may be assumed that no errors have been made either in making the multiplications or adding the columns.

The check sum as a check in correcting to the means.—After the values for ΣX_2^2, ΣX_2X_3, etc., have all been computed as indicated in Table 87, the process of making the corrections to get the values Σx_2^2, Σx_2x_3, etc., may be organized in regular fashion and checked by the check sum, as shown in Table 88.

TABLE 88

Calculation of Product-sums Corrected to Departures from Means, With Check Sum

	X_2	X_3	X_4	X_1	Σx	Line
Sums	134.	2177.	1376.	1377.	5064.	1
Means	10.30769	167.46154	105.84615	105.92308	389.53846	2
Extensions with X_2	1,994.00	25,315.00	14,158.00	14,224.00	55,691.00	3
Corrections	1,381.23	22,439.84	14,183.38	14,193.69	52,198.14	4
Extensions with x_2	612.77	2,875.16	−25.38	30.31	3,492.86	5
Extensions with X_3	....	409,443.00	233,148.00	235,519.00	903,425.00	6
Corrections	....	364,563.77	230,427.08	230,594.54	848,025.23	7
Extensions with x_3	....	44,879.23	2,720.92	4,924.46	55,399.77	8
Extensions with X_4	....	....	146,738.00	145,923.00	539,967.00	9
Corrections	....	....	145,644.30	145,750.15	536,004.90	10
Extensions with x_4	....	....	1,093.70	172.85	3,962.10	11
Extensions with X_1	....	....	....	146,855.00	542,521.00	12
Corrections	....	....	....	145,856.08	536,394.48	13
Extensions with x_1	....	....	....	998.92	6,126.52	14

The first line in Table 88 gives the sums of each of the variables, including the check sum. Dividing by the number of observations (13 in this case), gives the mean for each variable, as entered in the second line. Again, the entries for the first four columns total to equal the entry in the Σ column, checking the division.

The sums from the "extensions with X_2," of Table 87, are next entered in line 3. The sums for each variable, in line 1, are next multiplied by the mean of X_2 (10.30769), and the products entered in the corresponding column in line 4. Subtracting the entries in line 4 from those in line 3 gives the values which are entered in line 5. These values are the extensions, corrected for departures from the means.

In column X_3, for example, the entry in line 3 is ΣX_2X_3; and the entry in line 4 is ΣX_3M_2.

The entry in line 5, then, is $\Sigma X_2X_3 - \Sigma X_3M_2$

$$= \Sigma X_2X_3 - nM_3M_2$$

$$= \Sigma x_2x_3$$

Again, the values in the first four columns add to the same as the value in the check-sum column, verifying the work.

The rest of the table is entered in similar fashion. Lines 6, 9, and 12 are the extensions with X_3, X_4, and X_1, from Table 87. Lines 7, 10, and 13 are the values in the corresponding columns of line 1, multiplied by M_3, M_4, and M_1, respectively (from line 2). Lines 8, 11, and 14, obtained by subtracting the items in lines 7, 10, and 13 from those in 6, 9, and 12, show the values corrected for departures from the means.

In verifying the sum of the other entries in line 8 by the check sum, the item Σx_2x_3 must be included, from column X_3, line 5, before comparing with the check-sum; in checking line 11, Σx_2x_4 and Σx_3x_4, from column X_4, lines 5 and 8, must be included; and in checking line 14, the values Σx_1x_2, Σx_1x_3, and Σx_1x_4, from column X_1, lines 5, 8, and 11, must all be included. For line 11, the other items add to 3,962.11, as against the check sum of 3,962.10; and for line 14, they add to 6,126.55, as against the check sum of 6,126.52. In both cases the discrepancies are so small as to be readily due to raising and lowering in the last column, and, therefore, may be disregarded.

Lines 5, 8, 11, and 14 now give the values required to determine the regression coefficients by simultaneous solution, according to equations (38).

The check sum as a check in solving the normal equations.—The solution of the simultaneous equations by the Doolittle method has already been illustrated in Chapter 12, page 168. The check sum may be used to verify each step in the computation, as shown in Table 89.

TABLE 89

SOLUTION OF NORMAL EQUATIONS BY THE DOOLITTLE METHOD, WITH CHECK SUM

Line	X_2	X_3	X_4	X_1	Σx
I	612.77	2,875.16	−25.38	30.31	3,492.86
I′	−1.00000	−4.69207	0.04142	−0.04946	−5.70011
II	(2,875.16)	44,879.23	2,720.92	4,924.46	55,399.77
(−4.69206)(I)	(−2,875.16)	−13,490.45	119.08	−142.22	−16,388.74
Σ_2	....	31,388.78	2,840.00	4,782.24	39,011.03
II′	....	−1.00000	−0.09048	−0.15236	−1.24283
III	(−25.38)	(2,720.92)	1,093.70	172.85	3,962.10
(0.04142) (I)	(25.38)	(119.08)	−1.05	1.26	144.67
(−0.09048)(Σ_2)	....	(−2,840.00)	−256.96	−432.70	−3,529.72
Σ_3	....	....	835.69	−258.59	577.05
III′	....	....	−1.00000	0.30944	−0.69056

BACK SOLUTION

$b_{12\cdot34}$	$b_{13\cdot24}$	$b_{14\cdot23}$
0.04946	0.15236	−0.30944
−0.01282	0.02800	
−0.84626		−0.30944
	0.18036	
−0.80962		

The values from line 5 of Table 88, including the check sum, are entered as line I of Table 89. Each item is divided by the first item of the line, with its sign changed (-612.77). The quotients are entered as line I′. The sum of the first four items checks to the value in the last column, the check sum.

The values from line 8 of Table 88 are entered as line II of Table 89, beginning with column X_3 (the values enclosed in parentheses will be explained later). Line I is next multiplied by the value in column X_3 of line I′ (-4.69207), and the products entered in the corresponding columns below line II. These two lines are then summed, giving line Σ_2. These operations are now verified by adding the items of line Σ_2 in columns X_3, X_4, and X_1, and comparing the sum with the check sum in column Σ. The three values add to 39,011.02, agreeing to 0.01 with the check sum, 39,011.03.

The values in line Σ_2 are next divided by the value in column X_3, with its sign changed ($-31{,}388.78$). The quotients are entered as line II′. Again the check sum verifies the computation.

The values from line 11, Table 88, are then entered as line III, beginning with column X_4. (Again disregard the figures in parentheses.) Line I is multiplied by the value in column X_4 of line I′ (0.04142), and the products entered in the corresponding columns below line III; and line Σ_2 is multiplied by the value in column X_4 of line II′, and the products entered in the corresponding columns in the next line. Line III and the two following lines are then summed, giving line Σ_3. Again the check sum verifies the work. The values in line Σ_3 (before the check sum) add to 577.10, which agrees to 0.05 with the check sum of 577.05.

The values in lines I′, II′, and III′ of column X_1, *with the signs changed*, are then entered at the foot of columns X_2, X_3, and X_4 (designated here $b_{12.34}$, $b_{13.24}$, and $b_{14.23}$). The value at the foot of the X_4 column, -0.30944, is the value for $b_{14.23}$. The item in column X_4, line I′ (0.04142) is then multiplied by the last of these values (-0.30944), and the product (-0.01282) entered in the X_2 column; and the item in column X_4, line II′ (-0.09048) is also multiplied by -0.30944, and the product entered in the X_3 column. The two entries at the foot of the X_3 column are then added, giving 0.18036 as the value for $b_{13.24}$. The item in column X_3, line I′ (-4.69207) is then multiplied by 0.18036, and the product (-0.84626) entered below the other two entries at the foot of the X_2 column. The sum of these three entries, -0.80962, is then the value for $b_{12.34}$.

The way the check sum works in checking the operations may be seen by filling in the missing spaces in Table 89, as indicated by the entries enclosed in parentheses. Thus in line II, the first item, 2,875.16, is the same item, Σx_2x_3, as appears in line I, column X_3. If when line I had been multiplied by -4.69207, the operation had included the X_2 column also, the product would have been $-2{,}875.16$, or exactly the same as the item, in line I, column X_3, with the sign changed. This value, entered below line II in column X_3, exactly cancels the previous value when the two lines are added, leaving line II still the same.

Similarly, the values -25.38 and 2,720.92, from lines I and II of column X_4, may be entered in parentheses, in columns X_2 and X_3 of line III. If the previous operations had been carried out in full, below them would appear 25.38 in column X_2 (column X_4, line I, times -1), and 119.08 and $-2{,}840.00$ [(column X_4 line I) (-4.69206), and column X_4, line Σ_2, times -1]. When the three lines are totaled to give line Σ_3, the items exactly cancel out, as before.

It should be noted that when all the items are entered in each line, including those in parentheses, the sum of the items in columns X_2 to X_1 exactly equals, line by line, the item in column Σx. For that reason, if any error is found when one of the Σ lines is reached, the line in which the error occurred can be determined by adding the items line by line, and verifying the totals against the individual check sums. To do this it is not necessary to enter the missing items, as has been done in Table 89 (in parentheses); instead, the items left out can be picked out by going up the columns for the particular variable concerned. Thus all the missing terms for line III (extensions for X_4) and the next two lines appear above in the X_4 column. Once the location of the missing items in the previous work has been learned, they can be used to verify the computations line by line, and any error readily located.

The "back solution" is simply the solution, in regular form, of lines III′, II′ and I′ for b_4, b_3, and b_2. Thus line III′, if written out, is

$$-b_4 = 0.30944$$

Hence $b_4 = -0.30944$, the value at the foot of column X_4. Similarly, line II′ written out, becomes

$$-b_3 - 0.09048\, b_4 = -0.15236$$

Substituting the above value for b_4, and rearranging

$$\begin{aligned} b_3 &= 0.15236 - (0.09048)(-0.30944) \\ &= 0.15236 + 0.02800 \end{aligned}$$

These last two values are the same as shown at the foot of column X_3, hence $b_3 = 0.18036$.

Similarly line I′, when written out in full

$$-b_2 - 4.69207\, b_3 + 0.04142\, b_4 = -0.04946$$

Substituting values for b_3 and b_4, and rearranging

$$\begin{aligned} b_2 &= 0.04946 + (0.04142)(-0.30944) + (-4.69207)(0.18036) \\ &= 0.04946 - 0.01282 - 0.84626 = -0.80962 \end{aligned}$$

exactly as shown at the foot of column X_2.

Having computed the values of the three regression coefficients, the final steps are (*a*) to check those values by substituting them in the *last* equation (line III, in full); (*b*) to compute the coefficient of multiple correlation; and (*c*) to compute the constant $a_{1\cdot 234}$ for the regression equation. These steps are all shown in Table 90.

TABLE 90

FINAL STEPS IN SOLUTION OF MULTIPLE CORRELATION PROBLEM

Variable	Regression coefficient (1)	Equation III (2)	Check (3)	Equation X_1 (4)	Computation of R^2 (5)	Means (6)	Computation of a (7)
X_2	−0.80962	−25.38	20.55	30.31	−24.54	10.308	−7.86
X_3	0.18036	2720.92	490.75	4924.46	888.18	167.462	30.19
X_4	−0.30944	1093.70	−338.43	172.85	−53.49	105.846	−32.77
Sums	....	172.85	172.87	998.92	810.15	105.92	10.44

The first operation in Table 90 is the final checking of the entire solution, including the back solution. This is done by substituting the values found for the b's in the *last* equation of the normal equations. For this problem, that equation is:

$$\Sigma(x_2x_4)b_2 + \Sigma(x_3x_4)b_3 + \Sigma(x_4^2)b_4 = \Sigma x_1$$

The values of the 3 b's are entered in column (1) of the table, and the values of the corresponding coefficients of the unknowns, such as $\Sigma(x_2x_4)$ etc., are entered in column (2). The product of each b with its coefficient is then computed, and entered in column (3). These add to 172.87, checking satisfactorily with the value of $\Sigma(x_1^2)$, 172.85, as shown at the foot of column (2).

The computation of the coefficient of multiple correlation, according to equation (46)

$$R^2_{1.234} = \frac{b_2(\Sigma x_1x_2) + b_3(\Sigma x_1x_3) + b_4(\Sigma x_1x_4)}{\Sigma(x_1^2)}$$

is shown in tabular form in columns (4) and (5).

The values (Σx_1x_2), etc., as shown in Table 88, lines 5, 8, and 11 of column X_1, and $\Sigma(x_1^2)$, shown in line 14, are entered in column 4 of Table 90. Each product sum is multiplied by the corresponding b, shown in column (1), and the products entered in column (5). The sum of these products is then the numerator of the fraction in equation (46). The computation is then readily completed:

$$R^2_{1.234} = \frac{810.15}{998.92} = 0.8110$$

$$R = 0.9006. \quad \text{With } n = 13, \text{ and } m = 4,$$

$$\overline{R}^2 = 1 - (1 - 0.811)\tfrac{12}{9} = 0.784, \text{ and } \overline{R} = 0.86$$

The standard error may also be readily computed

$$\Sigma(x_1^2) = n\sigma_1^2 = 998.92$$

$$\Sigma[(x'_1)^2] = n\sigma_{x'}^2 = \underline{810.15}$$

then since $n\sigma_1^2 - n\sigma_{x'}^2 = n\sigma_z^2$

$$\Sigma(z^2) = n\sigma_z^2 = 188.77$$

Since there are 13 cases and 3 variables,

$$\overline{S}^2_{1.234} = \frac{n\sigma_z^2}{n - m} = \frac{188.77}{9} = 20.97$$

and

$$\overline{S}_{1.234} = 4.58$$

The a for the regression equation is next computed. Using equation (39)

$$a_{1.234} = M_1 - b_2M_2 - b_3M_3 - b_4M_4$$

the work may be arranged in tabular order as shown in columns (6) and (7) of Table 90. The means, from line 2 of Table 88, are entered in column (6), then multiplied by their respective b's, and the products entered in column (7). To complete the computation, following equation (39), the sum of this column is then subtracted from the mean of X_1.

$$a_{1.234} = 105.92 - 10.44 = 95.48$$

This completes the computation of all the linear multiple correlation constants. The results may be summarized as follows:

$$X'_1 = 95.48 - 0.810X_2 + 0.180X_3 - 0.309X_4$$

$$\overline{R}_{1.234} = 0.86$$

$$\overline{S}_{1.234} = 4.58$$

Tables 87, 88, 89, 90, and the computations following 90, have shown every arithmetic step in obtaining these results, arranged in the most convenient form for ready computation and checking. For problems involving large numbers of observations, the methods of computing the extensions, such as ΣX_2^2 and ΣX_1X_2, which are shown in Tables 85 and 86, may be used in place of the individual-item method shown in Table 87; but, thereafter, the work is the same. The check sum may be carried through in computations like those shown in Table 86 just as readily as in the longer method, thus providing a complete check on all the tabulating, multiplying, and adding.

In solving the equations in actual practice, only the items that are not enclosed in parentheses in Table 89 would be entered. Table 91 illustrates this same form of solution for a six-variable problem. It is carried out step by step, just as was Table 89. A slightly different notation is used, but the procedure through line III′ is the same. Then following line IV, line IV-1 is obtained by multiplying line I by -0.256900, the coefficient in line I′, column X_5; line IV-2 by multiplying line Σ_2 by -0.414640, the value in line II′, column X_5; and line IV-3 by multiplying line Σ_3 by -0.168107, the value in line III′, column X_5. A similar regular order is followed at the next step of the process, multiplying lines I, Σ_2, Σ_3, and Σ_4, by the coefficients in column X_6, lines I′, II′, III′, and IV′. Table 91 also illustrates the calculation of the back solution, the final check on the values of the b's by substituting in the last equation (eq. V), and the computation of R.

Standard Errors of Partial Regression Coefficients.—The computation of standard errors of net regression coefficients by equation (74), as discussed in Chapter 18, may be much simplified by the following procedure:

For three independent variables, set up the normal equations

$$\Sigma(x_2^2)c_{22} + \Sigma(x_2x_3)c_{23} + \Sigma(x_2x_4)c_{24} = 1$$

$$\Sigma(x_2x_3)c_{22} + \Sigma(x_3^2)c_{23} + \Sigma(x_3x_4)c_{24} = 0$$

$$\Sigma(x_2x_4)c_{22} + \Sigma(x_3x_4)c_{23} + \Sigma(x_4^2)c_{24} = 0$$

Solve simultaneously to obtain the values for c_{22}, c_{23}, and c_{24}. Then set up exactly the same set of equations, with c_{32}, c_{33}, and c_{34} as the unknowns, and with 0, 1, and 0 to the right of the equal signs, in the first, second, and third equations, respectively, and solve. Then set up again, with c_{42}, c_{43}, and c_{44} as the unknowns, and with 0, 0, and 1 to the right of the equal signs and solve again. The standard errors of the regression coefficients may then be found by the following equations:

$$\left.\begin{aligned} \sigma_{b_{12.34}} &= \overline{S}_{1.234}\sqrt{c_{22}} \\ \sigma_{b_{13.24}} &= \overline{S}_{1.234}\sqrt{c_{33}} \\ \sigma_{b_{14.23}} &= \overline{S}_{1.234}\sqrt{c_{44}} \end{aligned}\right\} \qquad (91)$$

TABLE 91

DOOLITTLE SOLUTION OF NORMAL EQUATIONS FOR SIX VARIABLES

Line Designation	Column Designation						
	X_2	X_3	X_4	X_5	X_6	X_1	Σx
EQUATIONS TO BE SOLVED							
Eq. I	100.00	23.32	19.86	25.69	10.64	40.17	219.68
Eq. II	(23.32)	100.00	17.47	45.20	21.39	60.03	267.41
Eq. III	(19.86)	(17.47)	100.00	26.28	0.33	23.79	187.73
Eq. IV	(25.69)	(45.20)	(26.28)	100.00	29.89	68.07	295.13
Eq. V	(10.64)	(21.39)	(0.33)	(29.89)	100.00	35.53	197.78
FRONT SOLUTION							
I	100.0000	23.3200	19.8600	25.6900	10.6400	40.1700	219.6800
I′	−1.000000	−0.233200	−0.198600	−0.256900	−0.106400	−0.401700	−2.196800
II	....	100.0000	17.4700	45.2000	21.3900	60.0300	267.4100
II-1	....	−5.4382	−4.6314	−5.9909	−2.4812	−9.3676	−51.2294
Σ_2	....	94.5618	12.8386	39.2091	18.9088	50.6624	216.1806
II′	....	−1.000000	−0.135769	−0.414640	−0.199962	−0.535760	−2.286130
III	....	....	100.0000	26.2800	0.3300	23.7900	187.7300
III-1	....	....	−3.9442	−5.1020	−2.1131	−7.9778	−43.6284
III-2	....	....	−1.7431	−5.3234	−2.5672	−6.8784	−29.3506
Σ_3	....	....	94.3127	15.8546	−4.3503	8.9338	114.7510
III′	....	....	−1.000000	−0.168107	+0.046126	−0.094725	−1.216706
IV	....	....	....	100.0000	29.8900	68.0700	295.1300
IV-1	....	....	....	−6.5998	−2.7334	−10.3197	−56.4358
IV-2	....	....	....	−16.2577	−7.8403	−21.0067	−89.6371
IV-3	....	....	....	−2.6653	+0.7313	−1.5018	−19.2904
Σ_4	....	....	....	74.4772	20.0476	35.2418	129.7667
IV′	....	....	....	−1.000000	−0.269178	−0.473189	−1.742367
V	....	....	....	....	100.0000	35.5300	197.7800
V-1	....	....	....	....	−1.1321	−4.2741	−23.3740
V-2	....	....	....	....	−3.7810	−10.1306	−43.2279
V-3	....	....	....	....	−0.2007	+0.4121	+5.2930
V-4	....	....	....	....	−5.3964	−9.4863	−34.9303
Σ_5	....	....	....	....	89.4898	12.0511	101.5408
V′	....	....	....	....	−1.00000	−0.134664	−1.134664

Back Solution					Eq. V	Check	Eq. X_1	Computation of R^2
$b_{12.3456}$	$b_{13.2456}$	$b_{14.2356}$	$b_{15.2346}$	$b_{16.2345}$				
+0.401700	+0.535759	+0.094725	+0.473189	+0.134664				
−0.014328	−0.026928	+0.006212	−0.036249	+0.134664	100.00	13.4664	35.53	4.7846
−0.112250	−0.181173	−0.073453	+0.436940		29.89	13.0601	68.07	29.7425
−0.005458	−0.003731	+0.027484			0.033	0.0091	23.79	0.6538
−0.075540	+0.323927				21.39	6.9288	60.03	19.4453
+0.194124					10.64	2.0655	40.17	7.7980
					Eq. V = 35.53	35.5299		62.4242

$$R^2 = \frac{62.4242}{\sigma_1^2} = \frac{62.4242}{100.00} \qquad R = \sqrt{0.624242} = 0.790090$$

It will be noted that, except for the values to the right of the equal sign, the coefficients of the equations are exactly the same as those required to obtain the values of $b_{12.34}$, $b_{13.24}$, and $b_{14.23}$. For that reason the values for c_{22}, c_{33}, and c_{44} may be most readily calculated by introducing as many new columns in the form of the Doolittle solution (Table 91) as there are independent factors, between the columns for X_1 and Σ. These columns will be

Line	Error b_2	Error b_3	Error b_4	Error b_5
(Eq. I)................	1	0	0	0
(Eq. II)................	0	1	0	0
(Eq. III)...............	0	0	1	0
(Eq. IV)................	0	0	0	1

etc.

These values can be included in the check sum, and the operations carried through for them just as for the other columns until the "back solution" to find the b's is reached. Then a separate "back solution" can be run for each set of "c" values, starting with the values in each "Error" column just as the back solution to find the b's started with the values in the X_1 column.[2]

Carrying out this calculation for the same problem worked through in Tables 87 to 90, the following values are secured:

$$c_{22} = 0.00233; \quad c_{33} = 0.000042; \quad c_{44} = 0.00196$$

Since $\bar{S}_{1.234} = 4.58$, the standard error of the b's may be readily calculated by equation (91)

$$\sigma_{b_{12.34}} = 4.58\sqrt{0.00233} = 0.221$$

$$\sigma_{b_{13.24}} = 4.58\sqrt{0.000042} = 0.030$$

$$\sigma_{b_{14.23}} = 4.58\sqrt{0.00196} = 0.203$$

The net regression coefficients may then be stated

$$b_{12.34} = -\ 0.810 \pm 0.221$$

$$b_{13.24} = \quad 0.180 \pm 0.030$$

$$b_{14.23} = -\ 0.309 \pm 0.203$$

Just as in the illustrations discussed in Chapter 18, some of the net regression coefficients are much more reliable than are others. Assuming the conditions of simple sampling to be fulfilled, there is some possibility that the regression for $b_{14.23}$ is really positive instead of negative; but there is only a very slight chance that $b_{12.34}$ is really positive, and it is almost a certainty that $b_{13.24}$ is really positive, and above 0.1. Referring to Table 90, it is interesting to note that variable X_3, for which the regression is most accurately determined, apparently contributed the most to the multiple correlation (note column 5). While this is frequently a fair guide to the most significant variables, it is not an unfailing one.

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[2] For an explanation of why this process and equation (91) gives the standard error of the b's, see Technical Note 14. For other uses of the "c" constants, see R. A. Fisher, Statistical Methods for Research Workers, 2nd ed., pages 132–138.

The computation of the standard errors of the net regression coefficients, by the method just presented, is not a difficult one. It should be made an integral part of every multiple correlation solution, so that not only will the regression coefficients be obtained, but the amount of confidence that can be placed in each value will be determined. Only if that is done can the regressions be interpreted with confidence.

Computing Residuals for Multiple Curvilinear Correlations.—Where there are a large number of individual observations, the average residual around the net regression line may be computed from group averages, instead of calculated for each individual observation as described in Chapter 14. This may save much time in calculating the average residuals to obtain the first approximation regression curves.

After the net linear regression coefficients are computed, the observations are thrown into groups with respect to the first independent factor, say X_2, and averages of each factor are computed for the records falling in each group. If there are four groups, for example, there will be four sets of averages.

Value of X_2	Average X_2	Average X_3	Average X_1
0– 9	M_{2-1}	M_{3-1}	M_{1-1}
10–19	M_{2-2}	M_{3-2}	M_{1-2}
20–29	M_{2-3}	M_{3-3}	M_{1-3}
30 and over	M_{2-4}	M_{3-4}	M_{1-4}

The average estimated value $M_{x'_1}$, may then be calculated for each group by substituting the means for that group in the regression equation. Thus for the first group

$$M_{x'} = a + b_2(M_{2-1}) + b_3(M_{3-1})$$

and

$$M_z = M_{1-1} - M_{x'}$$

In a similar manner the average residual may be calculated from the group averages for each of the other groups, and then plotted as a departure from the net regression line, as illustrated in Figure 35 of Chapter 14. After the computation is completed for X_2, the records may be reclassified with respect to X_3, new means calculated for each variable for each group, and the process continued just as for X_2. The same steps are carried out for each other independent variable in turn. This method may be used to determine the net residuals around curvilinear regressions fitted by mathematical curves just as well as for linear regressions.

Once the first set of free-hand approximation curves has been drawn, the remainder of the work has to be carried forward just as described in Chapter 14, as the average of values along a curve do not represent that curve in the same way that the average of values along a straight line may represent that line.

APPENDIX 2

TECHNICAL NOTES

Note 1. (Chapter 2).—The formula for estimating the standard error of an average is derived as follows:

Assume that we have N random samples of n observations each, all expressed in deviations, x, from the true mean of the variable, X, we are sampling. Designate the successive observations in each sample x', x'', x''', etc., as shown in the following tabular statement:

Observation	Sample 1	Sample 2	Sample 3	Sample N
1	x'	x'	x'	x'
2	x''	x''	x''	x''
3	x'''	x'''	x'''	x'''
4	x''''	x''''	x''''	x''''
....	....	....	....	....
....	....	....	....	....
....	....	....	....	....
n	x_n	x_n	x_n	x_n
	Σx_1	Σx_2	Σx_3	Σx_N

The first observations in the several samples (line 1) will have a standard deviation ($\sigma_{x'}$) which will tend to approach the true standard deviation (σ_x) of the universe of x's from which the observations are drawn. As N, the number of samples, becomes larger and larger, $\sigma_{x'}$ will tend to agree more and more closely with σ_x.

The second observations in the several samples (line 2), will have a standard deviation ($\sigma_{x''}$) which will also tend to agree more and more closely with the standard deviation of the universe (σ_x) as N becomes larger and larger.

Suppose now that the first and second observations in each sample are added. Let this sum be designated x_{1+2}. That is,

$$x' + x'' = x_{1+2}$$

The standard deviation of x_{1+2} is, by definition, $\dfrac{\Sigma(x_{1+2})^2}{N}$

But each

$$\begin{aligned}(x_{1+2})^2 &= (x' + x'')^2 \\ &= (x')^2 + 2x'x'' + (x'')^2\end{aligned}$$

Hence,

$$\Sigma(x_{1+2})^2 = \Sigma(x')^2 + 2\Sigma x'x'' + \Sigma(x'')^2$$

If the successive observations in each sample, x', x'', x''', are obtained by true random sampling, they will not be correlated with each other; that is, a large value for x' will be just as likely to be followed by a small value for x'' as by another large value.

The correlation of x' with x'' will tend to approach 0 as the number of samples, N, is increased. But if $r_{x'x''} = 0$, $\Sigma x'x''$ will equal 0. Hence, if the successive observations x' and x'' are uncorrelated, the last equation becomes

$$\Sigma(x_{1+2})^2 = \Sigma(x')^2 + \Sigma(x'')^2$$

Dividing by N

$$\sigma^2{}_{1+2} = \sigma^2{}_{x'} + \sigma^2{}_{x''}$$

But $\sigma_{x'}$ and $\sigma_{x''}$ both tend to equal σ_x

Hence

$$\sigma^2{}_{1+2} = 2\sigma^2{}_x, \text{ when } N \text{ is very large.}$$

Similarly, if the first three observations are added, the standard deviation of their sum, x_{1+2+3}, will be

$$\sigma^2{}_{1+2+3} = \sigma^2{}_{x'} + \sigma^2{}_{x''} + \sigma^2{}_{x'''}$$

$$= 3\sigma^2{}_x, \text{ when } N \text{ is very large.}$$

So if all n observations in each sample are added, the standard deviation of the sum, $x_{1+2+\ldots+n}$, will be

$$\sigma^2{}_{1+2+\ldots+n} = \sigma^2{}_{x'} + \sigma^2{}_{x''} + \ldots + \sigma^2{}_{x_n}$$

$$= n\sigma_x{}^2, \text{ when } N \text{ is large.}$$

If each observation, in each sample, and also the sum of all the observations in each sample, is divided through by n (the total number of observations in each sample), each x' will become x'/n, and each Σx will become $\Sigma x/n$. The standard deviation for the series of values x'/n for the first observation in each sample may then be computed

$$\text{each } \left(\frac{x'}{n}\right)^2 = \frac{(x')^2}{n^2}$$

and

$$\Sigma\left(\frac{x'}{n}\right)^2 = \frac{\Sigma(x')^2}{n^2} = \frac{N\sigma^2{}_{x'}}{n^2}$$

Dividing through by N,

$$\sigma^2{}_{x'/n} = \frac{\sigma^2{}_{x'}}{n^2}$$

and since $\sigma_{x'}$ tends to equal σ_x

$$\sigma^2{}_{x'/n} = \frac{\sigma_x{}^2}{n^2}, \text{ when } N \text{ is very large.}$$

The standard deviation for the second series of observations, x''/n, similarly will be

$$\sigma^2{}_{x''/n} = \frac{\sigma^2{}_{x''}}{n^2} = \frac{\sigma_x{}^2}{n^2}$$

The standard deviation of the sums of $x'/n + x''/n$ likewise will be

$$\sigma^2_{x'/n+x''/n} = \frac{\sigma_x^2}{n^2} + \frac{\sigma_x^2}{n^2} = \frac{2\sigma_x^2}{n^2}$$

So the standard deviation of the sums of all the n values x'/n to x_n/n will be

$$\sigma^2_{x'/n+x''/n+x'''/n+\ldots+x_n/n} = \frac{\sigma_x^2}{n^2} + \frac{\sigma_x^2}{n^2} + \ldots + \frac{\sigma_x^2}{n^2}$$

$$= \frac{n\sigma_x^2}{n^2}$$

$$= \frac{\sigma_x^2}{n}$$

But the sums $\Sigma\left(\frac{x}{n}\right)$ of all the values x'/n to x_n/n, that is the values $\frac{\Sigma x}{n}$, are the arithmetic averages for each sample, M_x. Hence, under conditions of simple sampling, the standard deviation of the arithmetic means of samples of n observations is given by the last equation, which may be written

$$\sigma_M^2 = \frac{\sigma_x^2}{n}, \text{ and hence, } \sigma_M = \frac{\sigma_x}{\sqrt{n}}$$

It should be noted that σ_x is the standard deviation of all the items in the universe, not the standard deviation of the items in a random sample. It has been shown by "Student" (Biometrika, vol. 6, p. 1, 1908) that the standard deviation, σ_s, of the items in a small random sample tends to be less than the standard deviation, σ_x, of all the items in the universe from which that sample was drawn, and that the true standard deviation of the universe may best be estimated from that of a small sample of n observations by use of the equation

$$\sigma_x^2 = \frac{\Sigma x^2}{n-1} = \frac{n\sigma_s^2}{n-1}$$

Substituting this value for σ_x^2 in the equation for σ_M^2

$$\sigma_M^2 = \left(\frac{n\sigma_s^2}{n-1}\right)\left(\frac{1}{n}\right)$$

$$= \frac{\sigma_s^2}{n-1}$$

and hence

$$\sigma_M = \frac{\sigma_s}{\sqrt{n-1}}$$

the equation given in the text.

It should be noted that an essential assumption made in deriving this formula is that the successive observations x', x'', etc., are not correlated with each other. In many types of economic problems, such as time series, for example, this assumption may be incorrect. In such cases the error formulas do not have the same usefulness as in cases where this assumption is approximately fulfilled.[1]

Note 2. (Chapter 5).—In fitting a straight line, the requirement is to determine, from the series of paired observations of X and Y, values for a and b in the equation

$$Y' = a + bX$$

which will make the sum of the squares of the residuals, $Y - Y'$, as small as possible.

The values whose sum is to be minimized are

$$\begin{aligned}(Y - Y')^2 &= (Y - a - bX)^2 \\ &= Y^2 + a^2 + b^2X^2 - 2aY - 2bYX + 2abX\end{aligned}$$

The sum of these values is therefore

$$\Sigma(Y - Y')^2 = \Sigma Y^2 + na^2 + b^2\Sigma(X^2) - 2a\Sigma(Y) - 2b\Sigma(YX) + 2ab\Sigma(X)$$

To determine the values of a and b which will make $\Sigma(Y - Y')^2$ a minimum, the partial derivatives with respect to a and b must be obtained and set equal to zero

$$\frac{\partial[\Sigma(Y - Y')^2]}{\partial a} = 2na - 2\Sigma(Y) + 2b\Sigma(X)$$

$$\frac{\partial[\Sigma(Y - Y')^2]}{\partial b} = 2b\Sigma(X^2) - 2\Sigma(YX) + 2a\Sigma(X)$$

Setting each equal to zero

$$2na - 2\Sigma(Y) + 2b\Sigma(X) = 0$$

$$na + \Sigma(X)b = \Sigma(Y) \qquad \text{(I)}$$

and

$$2b\Sigma(X^2) - 2\Sigma(YX) + 2a\Sigma(X) = 0$$

$$\Sigma(X)a + \Sigma(X^2)b = \Sigma(XY) \qquad \text{(II)}$$

Equations (I) and (II) are then the required normal equations, to be solved simultaneously, as given in Footnote 2 of Chapter 5.

Solving the normal equations for a and b, the steps are as follows:

$$na + \Sigma(X)b = \Sigma(Y) \qquad \text{(I)}$$

$$a + \frac{\Sigma(X)}{n}b = \frac{\Sigma(Y)}{n}$$

$$a = \frac{\Sigma(Y)}{n} - \frac{\Sigma(X)}{n}b$$

$$a = M_y - M_xb \qquad \text{(I')}$$

and

$$\Sigma(X)a + \Sigma(X^2)b = \Sigma(XY) \qquad \text{(II)}$$

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[1] This development follows that suggested in G. U. Yule, An Introduction to the Theory of Statistics, Chap. XVII, ¶10, pp. 344–345, and Chap. XI, ¶2, pp. 210–211, in the 6th edition, 1922.

substituting the value of a given above

$$\Sigma(X)M_y - \Sigma(X)M_x b + \Sigma(X^2)b = \Sigma(XY)$$

$$nM_xM_y - nM_x^2 b + \Sigma(X^2)b = \Sigma(XY)$$

$$\Sigma(X^2)b - nM_x^2 b = \Sigma(XY) - nM_xM_y$$

hence

$$b = \frac{\Sigma(XY) - nM_xM_y}{\Sigma(X^2) - nM_x^2} \tag{II'}$$

Equations (I′) and (II′) are then the equations given in the text as equations (10) and (9), to compute b and a.

Note 3. (**Chapter 6**).—In computing Σx^2 for a variable X, or Σxy for two variables X and Y, the values are not at all affected if some constant value is subtracted from each item of the series before the computations are made.

Let c represent the constant subtracted, and D_1 represent $X - c$.

Then

$$D_1^2 = (X - c)^2 = X^2 - 2Xc + c^2$$

$$\Sigma D_1^2 = \Sigma X^2 - 2c\Sigma X + nc^2$$

and

$$M_{D_1} = M_x - c$$

Now from equation (5)

$$\sigma_x = \sqrt{\frac{\Sigma(X^2)}{n} - M_x^2}$$

$$\Sigma x^2 = \Sigma X^2 - nM_x^2$$

Let

$$d = D_1 - M_{D_1}$$

Then

$$\begin{aligned}\Sigma d^2 &= \Sigma D_1^2 - nM^2_{D_1}\\ &= \Sigma X^2 - 2c\Sigma X + nc^2 - n(M_x - c)^2\\ &= \Sigma X^2 - 2cnM_x + nc^2 - nM_x^2 + 2ncM_x - nc^2\\ &= \Sigma X^2 - nM_x^2\end{aligned}$$

Hence

$$\Sigma d^2 = \Sigma x^2$$

By an exactly similar process, if $D_1 = X_1 - c_1$, and $D_2 = X_2 - c_2$, it can be proved that

$$\Sigma(d_1d_2) = \Sigma(x_1x_2)$$

Similarly, if any variable, X, is multiplied or divided by a constant, c, the effect will be to multiply or divide Σx^2 by c^2, and Σxy by c. Hence, where X is divided by c, σ_x and b_{xy} will be divided by c, but b_{yx} will be multiplied by c.

For

$$\Sigma\frac{X}{c} = (\Sigma X)\frac{1}{c};$$

and

$$M_{x/c} = (M_x)\left(\frac{1}{c}\right)$$

And

$$\Sigma\left(\frac{X}{c}\right)^2 = \Sigma(X^2)\frac{1}{c^2}$$

Hence

$$\Sigma\left(\frac{X}{c}\right)^2 - nM^2_{x/c} = \Sigma(X^2)\frac{1}{c^2} - n(M_x^2)\left(\frac{1}{c^2}\right)$$

$$= \frac{\Sigma X^2 - nM_x^2}{c^2}$$

$$= \frac{\Sigma x^2}{c^2}$$

So

$$\sigma_{x/c} = \frac{\sigma_x}{c}$$

Likewise,

$$\Sigma(X/c)(Y) = \Sigma(XY)^1/c$$

Hence

$$\Sigma(X/c)(Y) - nM_{x/c}M_y = \Sigma(XY)^1/c - n(M_x)(^1/c)(M_y)$$

$$= \frac{\Sigma XY - nM_xM_y}{c}$$

$$= \Sigma xy/c$$

$$b_{y(x/c)} = \frac{\Sigma(x/c)(y)}{\Sigma(x/c)^2} = \frac{\Sigma xy/c}{\Sigma(x^2/c^2)} = \frac{\Sigma xy}{\Sigma x^2/c} = b_{yx}c$$

Similarly,

$$b_{(x/c)y} = \frac{\Sigma(x/c)(y)}{\Sigma y^2} = \frac{\Sigma xy/c}{\Sigma y^2} = b_{xy}/c$$

Note 4. (Chapter 7).—To prove for a very simple case that r_{yx}^2 measures the proportion of variance in Y explained by X. Let a, b, c, etc., be series of variables with $\sigma_a = \sigma_b = \sigma_c$, and with all inter-correlations such as r_{ab}, r_{ac}, etc., $= 0$.

Let

$$Y = a + b + c$$

$$X = a + b$$

Then

$$r_{yx}^2 = \frac{p_{yx}^2}{\sigma_x^2\sigma_y^2}$$

(Here the symbol p_{yx} is used to represent $\Sigma yx/n$.)

$$\text{each } (y)(x) = (a + b + c)(a + b)$$

$$= a^2 + 2ab + ac + b^2 + bc$$

Since

$$\Sigma(ab),\ \Sigma(ac),\ \Sigma(bc) = 0$$

$$\Sigma(y)(x) = \Sigma a^2 + \Sigma b^2$$

Similarly,

$$(y^2) = (a + b + c)^2$$

$$= a^2 + 2ab + 2ac + b^2 + 2bc + c^2$$

$$\Sigma(y^2) = \Sigma a^2 + \Sigma b^2 + \Sigma c^2$$

By similar proof,

$$\Sigma(x)^2 = \Sigma a^2 + \Sigma b^2$$

Hence

$$r_{yx}^2 = \frac{(\sigma_a^2 + \sigma_b^2)^2}{(\sigma_a^2 + \sigma_b^2 + \sigma_c^2)(\sigma_a^2 + \sigma_b^2)}$$

$$= \frac{\sigma_a^2 + \sigma_b^2}{\sigma_a^2 + \sigma_b^2 + \sigma_c^2}$$

$$= 2/3 \text{ (since } \sigma_a = \sigma_b = \sigma_c)$$

Similar results will be obtained for other simple combinations of elements.

Note 5. (Chapter 7).—Instead of measuring the presence of correlation by comparing the standard deviation of the estimated values with the standard deviation of the actual, the amount of *absence* of correlation may be measured by comparing the standard deviation of the residuals—the standard error of estimate—with the original standard deviation. Thus in the horse feed problem, where the coefficient of correlation—r_{yx}—was found (before adjusting) to be equal to 3.47/7.92 or +0.44, for the straight line relation, the standard error of the residuals was equal to 7.13. If we express this in proportion to the original standard deviation, it gives us the ratio σ_z/σ_y, or, in this case, 7.13/7.92, which equals 0.90. This term has been given the name *coefficient of alienation*,[2] since it measures the *lack of correlation* in exactly the same way that the coefficient of correlation measures the presence of correlation.

In this particular case, the (unadjusted) coefficient of correlation is 0.44, and the coefficient of alienation is 0.90. The total of the two is considerably greater than 1.00. This, therefore, warns us that we cannot regard the coefficient of correlation as giving the *per cent* of correlation, or the coefficient of alienation as giving the *per cent* of absence of correlation. Except when one of the two values is equal to 0, the sum of the two will always be greater than unity.

If we look back to the values from which these coefficients were computed—the standard deviation of the original dependent series, the standard deviation of the estimated values, and the standard deviation of the residuals, it is easy to see why this is so. The standard deviation of the original values, $\sigma_y = 7.92$; the standard deviation of the estimated values, $\sigma_{y'}, = 3.47$; and the standard deviation of the residuals, $\sigma_z, = 7.13$. When we add the last two together, we find they equal more than the original standard deviation. Therefore, when we express each of them as a per cent of this original standard deviation, the sum of the two values is more than unity. But if we square these standard deviations, we find that $\sigma_y^2 = 62.73$, $\sigma_{y'}^2 = 12.04$, and $\sigma_z^2 = 50.84$. Here $\sigma^2_{y'}$ plus the σ_z^2, 12.04 plus 50.84, is equal to 62.88, practically identical with σ_y^2, 62.73.[3] This will always hold true, as each individual $Y' + z = Y$. It has already been proved (Note 1), that when Y' and z are not correlated, and $Y = Y' + z$, that then

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[2] Kelley, Truman L., Statistical method, pp. 173-175. 1924.

[3] The slight difference is due to dropping decimals in entering the Y' values.

$$\sigma^2_{y'} + \sigma_z^2 = \sigma_y^2$$

$\sigma_{y'} = \sigma_{yc}$

$\sigma_z = S_y$

which we have just observed to hold true in this case.

If we measure the amount of correlation by dividing $\sigma_{y'}^2$ by σ_y^2, and the lack of correlation by dividing σ_z^2 by σ_y^2, we will have two measures whose sum will always equal unity, so that when we know what one is, we can tell the other immediately. These values, $\sigma_{y'}^2/\sigma_y^2$, and σ_z^2/σ_y^2, are known respectively as the *coefficient of determination* and the *coefficient of non-determination*. They equal the square of the coefficient of correlation (r^2), and the square of the coefficient of alienation (k^2). In this case these values, $(0.44)^2$ and $(0.90)^2$, are 0.19 and 0.81, showing that (if the correction for the number of cases is ignored) 19 per cent of the variance in feed is associated with days worked, and 81 per cent is not so associated.

The adjustment of the coefficient of non-determination for the effect of small samples is indicated by the following formula:

$$k^2 = \frac{(n - 1)(\sigma_z^2/\sigma_y^2)}{n - m}$$

Applying this correlation, k^2 for the horse feed problem becomes 0.86, indicating that 86 per cent of the variance is independent of the days worked.

Since the several measures of correlation are all derived from the standard deviations of Y, Y', and z, certain mathematical relations always hold among the unadjusted coefficients, as is shown following:

$$\sigma_{y'}^2 + \sigma_z^2 = \sigma_y^2$$

$$\frac{\sigma_{y'}^2}{\sigma_y^2} + \frac{\sigma_z^2}{\sigma_y^2} = 1$$

$$r_{xy}^2 + k_{xy}^2 = 1 \text{ (k = the coefficient of alienation)}$$

or

$$d_{xy} + k_{xy}^2 = 1$$

$$r = \sqrt{1 - k^2}$$

also,

$$\frac{\sigma_z^2}{\sigma_y^2} = 1 - \frac{\sigma_{y'}^2}{\sigma_y^2} = 1 - r_{xy}^2$$

hence

$$\sigma_z^2 = (1 - r_{xy}^2)\,\sigma_y^2$$

or

$$\sigma_z = \sigma_y\sqrt{1 - r_{xy}^2}$$

This last equation is useful in estimating S_{yx}, the standard error in estimating Y from known values of X, when only the standard deviation of Y and the correlation of X with Y are known. As is shown in Chapter 8, the coefficient of correlation can be computed directly without first computing all the estimated values of Y (the Y' values) or without computing the individual residuals. When the correlation coefficient is thus computed, this last equation provides a short-cut to tell what the errors in estimating values of Y from the known values of X according to the straight line relation are likely to be.

Similarly, with curvilinear relations,

$$\sigma_{y''}^2 + \sigma_{z''}^2 = \sigma_y^2$$

and

$$\frac{\sigma_{y''}^2}{\sigma_y^2} + \frac{\sigma_{z''}^2}{\sigma_y^2} = 1$$

Hence,

$$\rho^2 + \frac{\sigma_{z''}^2}{\sigma_y^2} = 1$$

and

$$\rho^2 = 1 - \frac{\sigma_{z''}^2}{\sigma_y^2}$$

Note 6. (Chapter 12).—The normal equations, to determine the "best" regression values for two or more independent variables, are derived by exactly the same process as given in full in Note 2. Thus to determine the constants in the equation

$$X_1 = a + b_2X_2 + b_3X_3 + b_4X_4$$

The value to be made a minimum is

$$\Sigma(X_1 - a - b_2X_2 - b_3X_3 - b_4X_4)^2$$

Differentiating this with respect to a, b_2, b_3, and b_4, setting the partial derivatives equal to zero, and transposing, gives the normal equations, stated in terms of sums of X_1, X_2, etc. The equation may also be stated in terms of deviations from the means

$$x_1 = b_2x_2 + b_3x_3 + b_4x_4$$

This, by the same derivation, gives the normal equations as given in the text (equation 38). The constant, a, is then found separately by equation (39), which represents simply the separate solution of the first normal equation given by the first derivation,

$$\Sigma X_1 = na + \Sigma(X_2)b_2 + \Sigma(X_3)b_3 + \Sigma(X_4)b_4$$

Hence

$$a = \frac{\Sigma X_1}{n} - \frac{\Sigma X_2}{n}b_2 - \frac{\Sigma(X_3)}{n}b_3 - \frac{\Sigma(X_4)}{n}b_4$$

This is identical with equation (39).

Note 7. (Chapter 13.)—Coefficients of partial correlation are usually defined by the formula

$$r_{12.3} = \frac{r_{12} - r_{13}r_{23}}{\sqrt{1 - r_{13}^2}\sqrt{1 - r_{23}^2}}$$

For coefficients with more variables eliminated, such as $r_{12\cdot345}$, for example, this becomes

$$r_{12.345} = \frac{r_{12.34} - r_{15.34}r_{25.34}}{\sqrt{(1 - r^2_{15.34})(1 - r^2_{25.34})}}$$

To determine the coefficients with several factors held constant by this method involves a lengthy process of elimination, variable by variable; and for that reason the method presented in the text is preferred as shorter, simpler, and more readily subject to checking.

Note 8. (**Chapter 13**).—It can be proved that the coefficient of multiple determination (R^2) measures the percentage of variance ascribable to the several independent factors for certain simple cases. Thus assume four variables, A, B, C, D, with all intercorrelations equal to 0, and all σ equal. Let $Y = A + B + C$. Then correlate Y with A, B, and D. The regression equation will work out

$$Y = a + A + B[+0(D)]$$

Computing $R_{Y.ABD}$ by equation (46),

$$R^2_{Y.ABD} = \frac{b_{ya.bd}\Sigma ya + b_{yb.ad}\Sigma yb + b_{yd.ab}\Sigma yd}{\Sigma y^2}$$

Each

$$(y)(a) = (a + b + c)(a) = a^2 + ab + ac$$

$$\Sigma(ya) = \Sigma a^2 + \Sigma ab + \Sigma ac = \Sigma a^2 \text{ (Since } r_{ab} = r_{ac} = 0)$$

Similarly

$$\Sigma(yb) = \Sigma b^2; \ \Sigma(yd) = \Sigma d^2$$

And each

$$(y^2) = (a + b + c)^2 = a^2 + 2ab + b^2 + 2ac + 2bc + c^2$$

and

$$\Sigma(y^2) = \Sigma a^2 + \Sigma b^2 + \Sigma c^2$$

Hence

$$R^2_{Y.ABD} = \frac{(1)(\Sigma a^2) + 1(\Sigma b^2) + 0(\Sigma d^2)}{\Sigma a^2 + \Sigma b^2 + \Sigma c^2}$$

And since all σ are identical,

$$R^2_{Y.ABD} = 2/3$$

In this case, then, when Y, composed of three equally variable non-correlated elements, is correlated with two of those elements, and with one other equal element which is not represented in Y and which is not correlated with elements present in Y, the multiple determination of Y by the two elements (A and B) is found to be 2/3.

Similar results will be secured for other experimental cases which may be set up.

Note 9. (**Chapter 13**).—The coefficient of part correlation was first worked out by B. B. Smith, in collaboration with the present author, and was first published in "Correlation Theory and Method Applied to Agricultural Research," mimeographed publication, Bureau of Agricultural Economics, U. S. Dept. of Agriculture, August, 1926, pp. 57–60.

The formula for part correlation coefficients is derived as follows:

Let ${}_{12}r_{34}$ equal correlation of x_2 with $x_1 - b_{13.24}x_3 - b_{14.23}x_4$

Let

$$z = x_1 - b_2x_2 - b_3x_3 - b_4x_4$$

$$z + b_2x_2 = x_1 - b_3x_3 - b_4x_4$$

Then

$$({}_{12}r_{34})^2 = \frac{[\Sigma(z + b_2x_2)(x_2)]^2}{[\Sigma(z + b_2x_2)^2][\Sigma x_2^2]}$$

each

$$(z + b_2x_2)(x_2) = zx_2 + b_2x_2^2$$

$$\Sigma(z + b_2x_2)(x_2) = b_2\Sigma x_2^2 \qquad (r_{zx_2} = 0, \text{ hence } \Sigma z x_2 = 0)$$

each

$$(z + b_2x_2)^2 = z^2 + 2b_2x_2z + b_2^2x_2^2$$

$$\Sigma(z + b_2x_2)^2 = \Sigma z^2 + b_2^2\Sigma x_2^2$$

So

$$({}_{12}r_{34})^2 = \frac{b_2^2(\Sigma x_2^2)^2}{(\Sigma z^2 + b_2^2\Sigma x_2^2)(\Sigma x_2^2)} = \frac{b_2^2\Sigma x_2^2}{\Sigma z^2 + b_2^2\Sigma x_2^2}$$

$$= \frac{1}{\dfrac{\Sigma z^2}{b_2^2\Sigma x_2^2} + 1} = \frac{1}{\dfrac{n\sigma_1^2(1 - \bar{R}^2_{1.234})}{b_2^2 n\sigma_2^2} + 1}$$

$$= \frac{1}{1 + \dfrac{\sigma_1^2(1 - \bar{R}^2_{1.234})}{b_2^2\sigma_2^2}}$$

Note 10. (Chapter 13).—The coefficient of partial correlation, as used in biometric work, is employed to measure the correlation in an hypothetical universe, from which all variation due to changes in the eliminated factors has been excluded. This correlation may then be compared to the simple correlation of the two factors, to see whether the closeness of relation is improved or not by excluding the variation *in both variables* associated with other factors. With the coefficient of part correlation, on the contrary, all of the original variation in the independent factor is left in it, and only the dependent factor is adjusted.

Note 11. (Chapter 13). Separate determination.—Neither the coefficients of partial determination nor the coefficients of part determination equal, when totaled, the total determination of X_1 by X_2, X_3, and X_4. That is because both of these types of measures are computed on bases which change from variable to variable; hence their sums, when several are added together, have no mathematical significance. There is, however, a third type of coefficient, which parcels out among each of the several independent variables that part of the variation in the dependent variable which each one of them seems able to account for, when estimates of the dependent variable are made from all of the independent variables. To distinguish this type from the coefficients previously presented, they may be termed *coefficients of separate determination.*

Using $d_{12.34}$ to represent the separate determination of X_1 by X_2, when X_3 and X_4 are also considered, it may be computed by the formula

$$d_{12.34} = \left[\frac{b_{12.34}(\Sigma x_1x_2)}{\Sigma(x_1^2)}\right]\left[\frac{\bar{R}^2_{1.234}}{R^2_{1.234}}\right]$$

Each of these values has been used previously in computing the coefficient of multiple correlation, so the determination coefficient may be readily calculated.

When the several coefficients of separate determination are added together. their sum is equal to the coefficient of total determination, R^2. Comparing the last equation with equation (46) for R^2, it is readily seen that

$$\bar{R}^2_{1.234} = \left[\frac{b_{12.34}(\Sigma x_1x_2)}{\Sigma(x_1^2)} + \frac{b_{13.24}(\Sigma x_1x_3)}{\Sigma(x_1^2)} + \frac{b_{14.23}(\Sigma x_1x_4)}{\Sigma(x_1^2)}\right]\left[\frac{\bar{R}^2_{1.234}}{R^2_{1.234}}\right]$$

The three terms on the right are the three coefficients of separate determination, $d_{12\cdot34}$, $d_{13\cdot24}$, and $d_{14\cdot23}$. These coefficients are the simplest to compute of any of the three types which have been discussed, the computation of their values being readily made a part of the process of working out R and S.

Working out the last equation for acres and income, in the 4-variable problem, it becomes

$$d_{12.34} = \left[\frac{(1.20584)(0.71)}{(272.76)}\right]\left[\frac{0.806}{0.8366}\right] = \frac{0.690}{228.19} = 0.00302$$

The corresponding values are 0.630 for the separate determination of income by cows, and 0.171 for the separate determination of income by number of men.

The coefficients of "separate" determination are the easiest of all to compute, and have the further advantage of adding to a definite sum ($\bar{R}^2$), and hence being directly comparable one with another. The disadvantage in their use, however, is that under certain conditions the value of one or more coefficients will prove to be negative. Off-hand it seems difficult to explain how the "determination" of any variable can be less than nothing. (This result will be obtained whenever the gross or apparent correlation and the coefficient of net or partial regression, are of opposite sign.) The explanation is simple, however. Although the total variation in the estimates of the dependent variable is obtained by adding the contributions from the several independent variables, it does not follow that all variables will be influencing the estimate in the same direction at the same time—all tending to give low values when the actual value is low, or all tending to give high values when the actual value is high. It sometimes happens that one variable may tend to work counter to the other variables, usually preventing the final estimate from going so low as it otherwise would when the general effect is downward, and tending to keep it from going so high as it otherwise would when the others are forcing it up. It is under such conditions that negative coefficients of separate determination are obtained; they do not mean that the variable has no significance, but that its influence is usually exerted counter to the influence of other variables.

When there is very high inter-correlation between the several independent variables, the coefficients of separate determination may vary quite erratically, and hence become of little significance. Under such conditions other measures of the individual importance of the several factors will need to be employed.

While nothing is known of the sampling error involved in determining coefficients of separate determination, since they are computed from standard errors, product sums, and net regression coefficients, their standard error must be some function of the standard error of these other coefficients. It is under the conditions noted in the preceding paragraph that net regression coefficients have the least reliability, so it may be that a problem which fails to yield reasonable separate determination coefficients may also fail to yield reliable values for the other measures of determination.

There seems to be evidence that coefficients of separate determination are less stable, and more subject to random error, than any other measure of the importance of individual factors. On account of their ease of computation, they have been much used in the past; but it is doubtful how much confidence can be placed in them. For that reason it seems best to use the other measures, and discard this measure until its reliability has been more definitely determined.

The advantage of the beta coefficients over coefficients of separate determination may be shown mathematically.

The normal equations for determining the regression coefficients

$$\Sigma x_2^2 b_2 + \Sigma x_2 x_3 b_3 + \Sigma x_2 x_4 b_4 = \Sigma x_1 x_2$$

$$\Sigma x_2 x_3 b_2 + \Sigma x_3^2 b_3 + \Sigma x_3 x_4 b_4 = \Sigma x_1 x_3$$

$$\Sigma x_2 x_4 b_2 + \Sigma x_3 x_4 b_3 + \Sigma x_4^2 b_4 = \Sigma x_1 x_4$$

may also be written after dividing through by n, the number of cases, and dividing each line and column by the corresponding standard deviation. Solution of these equations, which are shown below, then gives the values for the partial (net) beta coefficients.

$$\beta_2 + r_{23}\beta_3 + r_{24}\beta_4 = r_{12}$$

$$r_{23}\beta_2 + \beta_3 + r_{34}\beta_4 = r_{13}$$

$$r_{24}\beta_2 + r_{34}\beta_3 + \beta_4 = r_{14}$$

For the first set of equations (the usual statement) the separate determination of X_1 by X_2, $d_{12\cdot34}$, is given by the equation

$$d_{12.34} = \frac{b_2 \Sigma x_1 x_2}{\Sigma x_1^2}\left(\text{i.e.,} = \frac{b_{12.34}\Sigma x_1 x_2}{\Sigma x_1^2}\right)$$

In terms of the values given in the second set of normal equations, the coefficient of separate determination would be computed

$$d_{12.34} = \beta_2 r_{12} (\text{i.e.,} = \beta_{12.34} r_{12})$$

Substituting the value for r_{12} given by the first equation of the second set, this becomes

$$d_{12.34} = \beta_2(\beta_2 + r_{23}\beta_3 + r_{24}\beta_4)$$

$$= \beta_2^2 + r_{23}\beta_2\beta_3 + r_{24}\beta_2\beta_4$$

Similarly,

$$d_{13.24} = \beta_3^2 + r_{23}\beta_2\beta_3 + r_{34}\beta_3\beta_4$$

and

$$d_{14.23} = \beta_4^2 + r_{24}\beta_2\beta_4 + r_{34}\beta_3\beta_4$$

It is evident from this that each coefficient of separate determination consists of one portion, β^2, which is, as Dr. Sewall Wright named it, the "direct determination" by that independent variable; plus (or minus) a pro-rated share of the joint determination of that variable with each other independent variable. Since $r_{23}\beta_2\beta_3$ contributes equally to both $d_{12.34}$ and $d_{13\cdot24}$, this "joint determination" is simply divided equally between both independent variables. As only the direct determination (β^2) can be said to reflect the *separate influence* of the particular independent variable, the further attempt to allocate or split up the joint influence is unsatisfactory. For that reason, the several betas (or their squares) seem the best measures of the separable importance of each variable, leaving the combined influence of variables acting jointly out of the distribution.

This explanation follows that developed by H. R. Tolley, and presented in full in the bulletin by F. F. Elliott, Adjusting Hog Production to Market Demand, Univ. of Ill. Agr. Expt. Sta. Bul. 293, 1927. See also Sewall Wright, Correlation and Causation, Jour. Agr. Research, Vol. XX, No. 7, pp. 557–575.

Note 12. (Chapter 13).—Given the multiple regression equation

$$X_1 = a + b_2X_2 + b_3X_3 + b_4X_4$$

let

$R_{(X_1-b_2X_2).3,\,4}$ represent the correlation

between

$[X_1 - b_2X_2]$, and X_3 and X_4.

To find the formula for this correlation:
Using departures from their means for all variables

$$z = x_1 - b_2x_2 - b_3x_3 - b_4x_4$$

hence

$$b_3x_3 + b_4x_4 = x_1 - b_2x_2 - z$$

The required correlation is therefore that between

$[x_1 - b_2x_2]$ and $[x_1 - b_2x_2 - z]$

From equation (27) it is equal to

$$\frac{\Sigma[(x_1 - b_2x_2)(x_1 - b_2x_2 - z)]}{\sqrt{\Sigma(x_1 - b_2x_2)^2\Sigma(x_1 - b_2x_2 - z)^2}}$$

Each

$$(x_1 - b_2x_2)(x_1 - b_2x_2 - z) = (x_1 - b_2x_2)^2 - zx_1 + b_1zx_2$$

and

$$\Sigma[(x_1 - b_2x_2)(x_1 - b_2x_2 - z)] = \Sigma(x_1 - b_2x_2)^2 - \Sigma zx_1 + b_1\Sigma zx_2$$

Since z is uncorrelated with x_2, $\Sigma zx_2 = 0$.
The value of Σzx_1 may be evaluated as follows:

$$x_1 = b_2x_2 + b_3x_3 + b_4x_4 + z$$

$$zx_1 = b_2zx_2 + b_3z_3 + b_4zx_4 + z^2$$

$$\Sigma zx_1 = b_2\Sigma zx_2 + b_3\Sigma zx_3 + b_4\Sigma zx_4 + \Sigma z^2$$

But

$$r_{z2} = r_{z3} = r_{z4} = 0;$$

hence

$$\Sigma zx_1 = \Sigma z^2$$

Therefore,

$$\Sigma[(x_1 - b_2x_2)(x_1 - b_2x_2 - z)] = \Sigma(x_1 - b_2x_2)^2 - \Sigma z^2$$

Similarly,

$$\Sigma(x_1 - b_2x_2 - z)^2$$

may be shown to equal

$$\Sigma(x_1 - b_2x_2)^2 - \Sigma z^2$$

Hence

$$R_{(X_1-b_2X_2).3,\,4} = \frac{\Sigma(x_1 - b_2x_2)^2 - \Sigma z^2}{\sqrt{\Sigma(x_1 - b_2x_2)^2}\sqrt{\Sigma(x_1 - b_2x_2)^2 - \Sigma z^2}}$$

And

$$\begin{aligned} R^2{}_{(X_1-b_2X_2).3,\,4} &= \frac{[\Sigma(x_1 - b_2x_2)^2 - \Sigma z^2]^2}{[\Sigma(x_1 - b_2x_2)^2][\Sigma(x_1 - b_2x_2)^2 - \Sigma z^2]} \\ &= \frac{\Sigma(x_1 - b_2x_2)^2 - \Sigma z^2}{\Sigma(x_1 - b_2x_2)^2} \\ &= 1 - \frac{\Sigma z^2}{\Sigma(x_1 - b_2x_2)^2} \\ &= 1 - \frac{\sigma_z{}^2}{\sigma_1{}^2 - 2b_2p_{12} + b_2{}^2x_2{}^2} \\ &= 1 - \frac{\sigma_1{}^2(1 - R^2{}_{1.234})}{\sigma_1{}^2 - 2b_2\dfrac{\Sigma x_1x_2}{n} + b_2{}^2\sigma_2{}^2} \end{aligned}$$

Note 13. (**Chapter 18.**)—Extensive experiments, soon to be published in detail, have shown that the following provisional formulas will give a rough approximation to the standard error of net regression curves. These formulas express the four points mentioned at the end of Chapter 18. In experimental work, when computed from the results shown by individual samples, these formulas have, on the average, successfully indicated the range within which the true regression curves lie 17 times out of 20 (using a range of twice the computed standard error). The proportion of very large errors, up to 5 or more times the computed standard errors, has been much larger than would be expected from a normal distribution of errors. Until more work has been done on the problem, therefore, it seems either that these preliminary formulas may leave out some essential element in occasional cases, or that the results of graphic free-hand curve fitting may show errors in exceptional samples out of all proportion to those ordinarily made. In either event, these formulas are only tentative.

The tentative formulas are as follows:

$$\left.\begin{array}{l}\text{Standard error of (the difference between the}\\ \text{reading for any point } X_a \text{ on a regression curve and}\\ \text{the reading for the point corresponding to the}\\ \text{mean value, } M_x\text{)}\end{array}\right\} = \sqrt{\frac{\bar{S}^2{}_{y\cdot f(x)}ux}{\sigma_x{}^2n_u}} \qquad \text{(I)}$$

$$\left.\begin{array}{l}\text{Standard error of (the difference between the}\\ \text{reading for any point } X_a \text{ on a net regression curve}\\ \text{for } X_1 = f(X_2)\text{, and the reading for the point cor-}\\ \text{responding to the mean value, } M_2\text{)}\end{array}\right\} = \sqrt{\frac{\bar{S}^2{}_{1.f(2,\,3,\,4\,\ldots)}ux_2}{\sigma_2{}^2n_u(1 - R^2{}_{2.34\,\ldots})}} \qquad \text{(II)}$$

Since several new symbols are introduced in these two equations to cover the points which have been enumerated, they will first be defined.

The symbols have exactly the same meaning for both equations, except for the additional term $(1 - R^2{}_{2.34})$ in equation (II) for regression curves determined by multiple correlation. The standard errors of estimate, $\bar{S}_{y\cdot f(x)}$, and $\bar{S}_{1\cdot f(2,\,3,\,4)}$ have the same meaning as defined in equations (24) and (64); whereas σ_x and σ_2 are the usual standard deviations of the independent variable. The new terms have the following meanings:

n_u represents the number of observations falling within some selected group interval of X or X_2, with X_a, the point for which the accuracy of the curve is to be determined, at the center. This interval must be taken large enough to include the observations which were taken into account in determining the shape of that portion of the curve, yet small enough not to take in observations whose values did not enter into the determination of that part of the curve.

u designates the range over which n_u is taken, stated in units of the independent variable X or X_2, and using the same units as those in which the standard deviation, σ_2, is stated. Thus if the standard deviation is in units of pounds or dollars, u is also stated in pounds or dollars.

The other term, x or x_2, has the same meaning as used previously—the deviation of the independent variable from the mean of that variable, stated in the same terms as the standard deviation is stated in. Thus for the point along the curve where the independent variable has the value X_a, $x = X_a - M_x$. There is this difference from the usual usage, however,—in equations (I) and (II) x and x_2 are to be taken as positive numbers without regard to sign.

The several steps in working out the reliability of a regression curve, and the meaning of the results, may be illustrated by applying these equations to one of the curves previously determined.

The reliability of the regression curve worked out for cotton yields in Chapter 8 may be tested by equation (I). The curve obtained (Figure 23), shows that with an average application of water, 1.97 feet, a yield of 328 pounds of cotton would probably be obtained, whereas with an application of 1.4 feet of water, a yield of 195 pounds would probably be obtained. Apparently reducing the application of water 0.57 foot would reduce the yield of cotton by 133 pounds. How accurate is this last conclusion?

Picking out the values necessary to compute the standard error according to equation (I), we have $\bar{S}_{y \cdot f(x)} = 80.7$ pounds, $\sigma_x = 0.73$ foot, and $x = 0.57$ foot. Noting that the average yield in the groups of from 1 to 1.4 feet of water and from 1.5 to 1.9 feet of water both had some influence in determining the position of the curve at 1.4 feet, we may let the interval for which we take the number of cases extend half-way into the upper group, or to 1.7, and an equal distance below 1.4, or to 1.1. The number of items for n_u, then, will include all the cases having 1.05 or more feet of water applied, and less than 1.75 feet applied, a range of 0.70 foot. The number of cases falling in this range (Table 31) is found to be 6; so $n_u = 6$, and $u = 0.70$. Substituting these several values in equation (I), the arithmetic is as follows:

Standard error of decrease of 121 pounds

$$= \sqrt{\frac{\bar{S}^2_{y \cdot f(x)} u x}{\sigma_x{}^2 n_u}} = \sqrt{\frac{(80.7)^2(0.70)(0.57)}{(0.73)^2(6)}} = \sqrt{\frac{(6523)(0.70)(0.57)}{(0.5348)(6)}} = \sqrt{811.11}$$

Standard error $= 28.5$

The difference of 133 pounds, therefore, has a standard error of 29 pounds. The statement that the reduction of 0.57 foot in water applied reduces yields by 133 pounds, therefore, really means that the reduction is probably between 104 and 162 pounds, but there is one chance in 20 that it is as little as 77 pounds, or as much as 189 pounds. (Table A need not be considered in computing these chances, unless the total number of observations in the problem, $n - m$, is less than 30. Then $n - m$ should be used to find the column to determine the probabilities from, rather than n_u. In this case, with $n - m = 11$, the conclusions are not quite so reliable as these statements indicate.)

It should be noted that the estimated range of error would not have been changed very greatly if a different interval had been used for u. Had the range from 1.25 to 1.45 been taken instead, u would have been 0.30 and n_u would have been 5. Substituting these values in equation (I) instead of the ones used previously, the standard error works out as somewhat lower than before. The greater the total number of observations, the less effect a change in u will have on the computed error—it is only the small number of cases and very irregular distribution that causes as considerable a difference as in this particular case—and even so, the indicated reliability is still of the same order.

To compute the range of error for the entire curve, we may pick out a number of selected points—say at each 0.2 foot of water—and work out the error for the reading at each of those points. The process may be shortened by noting that in equation (I), the values of the several terms remain unchanged for every point along the curve, with the exception of u, n_u, and x; and that if the same range is taken for u at each point, only the two other values are changed. Accordingly, equation (I) may be restated as follows:

$$\text{Standard error for } f(X_0) - f(X_M) = \sqrt{k\frac{x}{n_u}}; \text{ where } k = \frac{\bar{S}^2_{y \cdot f(x)}u}{\sigma_x^2} \qquad \text{(III)}$$

Since the value of k is the same for every point along the curve, it can be worked out once for all; then all that needs to be computed at each point is $\frac{x}{n_u}$, the product $k\frac{x}{n_u}$, and the square root of the product.

The work of applying this process to the cotton yield curve may be shown in tabular form. First the value of k must be worked out. Continuing to use the same range for u as before, 0.70 foot of water, the computation is as follows:

$$k = \frac{\bar{S}^2_{y \cdot f(x)}u}{\sigma_x^2} = \frac{(80.7)^2(0.70)}{(0.73)^2} = 8538$$

The next step is to enter X for each selected point, compute the value of x, and determine the value of n_u and of x/n_u; then multiply by k and extract the square root. These several steps are shown in Table 92.

The values of n_u are determined just as in the single case before, by taking all the cases falling within 0.35 foot above and 0.35 foot below the value of X selected. Thus for $X = 1.2$, there are 6 cases between 0.85 and under 1.55; while for $X = 2.4$, there are but 3 cases between 2.05 and below 2.75. The series of value in the n_u column add to considerably more than the total number of observations, since the range taken is such that there is considerable overlapping. This does not affect the final errors computed, however, since the unit selected for u tends to have no effect on the size of the computed error.

The importance of the errors shown in the last column of Table 92 may be judged by comparing them to the values to which they apply—the difference between the estimated cotton yield for the several values of X, and the yield estimated for the mean value of X. Table 93 shows this comparison.

TABLE 92

Computing the Standard Errors for Points Along a Regression Curve

X	x	n_u	$\frac{x}{n_u}$	(Error)^2 $k\left(\frac{x}{n_u}\right)$	Error $\sqrt{k\frac{x}{n_u}}$
1.2	0.77	6	0.12833	1095.71	33.1
1.4	0.57	6	0.09500	811.11	28.7
1.6	0.37	8	0.04625	384.89	19.6
1.8	0.17	6	0.02833	241.91	15.5
2.0	0.03	5	0.00600	51.23	7.1
2.2	0.23	4	0.05750	490.94	22.1
2.4	0.43	3	0.14333	1223.78	35.0
3.5	1.53	2	0.76500	6531.57	80.8

The standard errors will have to be interpreted with respect to the total number of observations, adjusted by m. For this problem, $m = 3$, so Table A should be entered with 12. Interpolating, we find that for such samples a departure of more than one standard error from the true value is likely to occur 34 times out of 100, and a departure of more than twice the standard error is likely to occur about 8 times out of 100 (as compared to less than 5 times for a very large sample). To reduce the

TABLE 93

Deviation of Points on a Regression Curve from the Value for the Mean, and the Standard Errors

X	$f(X)$	$f(X) - f(X_M)$	Standard errors *	2.25 (standard error) †
1.2	145	−183	±33	±75
1.4	195	−133	±29	±65
1.6	248	− 80	±20	±44
1.8	293	− 35	±16	±35
1.9	328	0		
2.0	335	7	± 7	±16
2.2	376	48	±22	±50
2.4	414	86	±35	±79
3.5	543	215	±81	±182

* From Table 92.

† For a case where $n - m = 11$, range within which the probabilities are 0.95 that the true curve lies.

probability of the true differences in yield lying beyond the observed differences in yield to 5 chances out of 100 (1 out of 20) it is necessary to add about ± 2.25 times the standard error to the observed differences. This value is accordingly entered in the final column of Table 93.

Just as with a regression line, the value of Y corresponding to the mean value of X is not exactly certain. No special sampling study has been made of regression curves in this connection. It would hardly be correct to apply equation (70) directly to this, as the central value for a free-hand regression curve is not as definitely determined as for a straight regression line. Accordingly, the errors may only be interpreted with respect to differences from the mean value, rather than with respect to actual values.

The regression curve with its computed standard error is shown in Fig. 68, together with the wider range to reduce the probability of error to 0.05. This chart indicates the interpretation which may be given to the computed errors. The inner borders indicate the range within which the relation probably lies, with 1 chance in 3 of the true relation falling outside this range; while the outer borders indicate the range within which the true curve probably falls, but possibly with a greater chance of falling outside this range than 1 in 20.

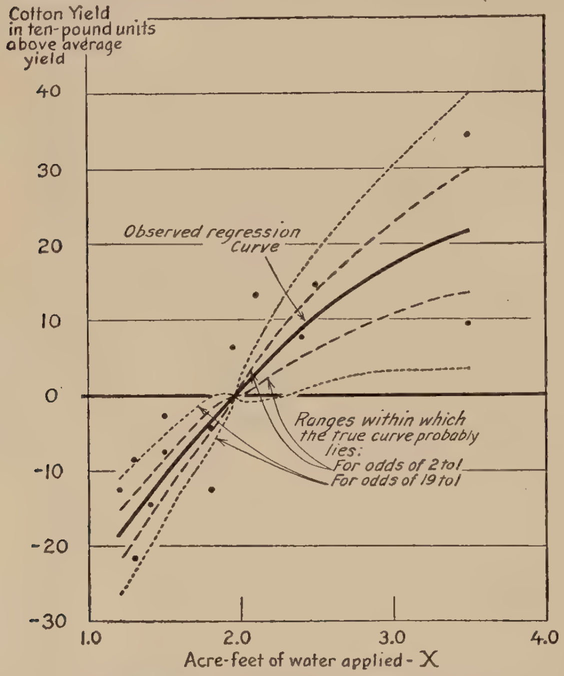

FIG. 68.—Curvilinear regression of cotton yield on irrigation water applied, and range within which the true relation probably lies.

It is now quite evident why the table showing the relation according to the regression curve, as set forth in Chapter 8, page 135, was not carried beyond 2.5 feet of water. In fact, it might be just as well not to carry it beyond 2.25 feet, to judge from Figure 68, for at 2.5 feet there is at least 1 chance out of 20 that the true difference in yield above the yield for the average water application differs from that estimated by nearly the difference shown in the estimate. The wide range of possible error for the true position of this curve reflects both the small number of observations upon which it is based, and the relatively low correlation shown by those observations. Even so, the computed range of error indicates what degree of reliance can be placed in the findings under these limiting conditions, and so makes the results of the analysis of more value than if we had no knowledge of their probable stability.

The process of working out the standard error for a net or partial regression curve is exactly the same as that just illustrated, except that equation (II) is used instead of (I). The computation may be broken into two steps just as illustrated for simple correlation, as follows:

$$\text{Standard error for } f(X_0) - f(X_M) = \sqrt{k'\frac{x_2}{n_u}}, \text{ where } k' = \frac{\bar{S}^2_{1.f(2,\,3,\,4)u}}{\sigma_2^2\,(1 - R^2_{2.34})} \qquad \text{(IV)}$$

Since the process of calculating the standard errors for a net regression curve is exactly the same as for a simple regression curve, equation (IV) will not be illustrated in detail.

In cases where the inter-correlations of the several independent variables are markedly curvilinear, $P^2_{2.34}$ should be used in equation (IV) instead of $R^2_{2.34}$. It is evident that it would require a considerable volume of computation to determine even the approximate limits for each net regression curve in a multiple correlation problem. It is also evident from the single illustration, however, that certain portions of a regression curve may be much less accurately determined than certain other portions. Until adequate methods are employed to judge the reliability of multiple curvilinear regressions, the statements as to the exact functional relation of the dependent factor to each independent factor will be of unknown reliability.

Note 14 (for Appendix on Methods of Computation.)—To prove that equation (91)

$$\sigma_{b_{13.24}} = \bar{S}_{1.234}\sqrt{c_{33}} \qquad (91)$$

gives the same value as that given by equation (74)

$$\sigma_{b_{13.24}} = \sqrt{\frac{\bar{S}^2_{1.234}}{\sigma_3^2(1 - R^2_{3.24})n}} \qquad (74)$$

For a problem in two independent variables, c_{33} is obtained by the simultaneous solution of the equations

$$\Sigma(x_2^2)c_{32} + \Sigma(x_2x_3)c_{33} = 0$$
$$\Sigma(x_2x_3)c_{32} + \Sigma(x_3^2)c_{33} = 1$$

Solving by the Doolittle method

$$\begin{aligned}
\Sigma(x_2^2)c_{32} + \Sigma(x_2x_3)c_{33} &= 0 \\
-c_{32} - \frac{\Sigma(x_2x_3)}{\Sigma x_2^2}c_{33} &= 0 \\
\hline
\Sigma(x_2x_3)c_{32} + \Sigma(x_3^2)c_{33} &= 1 \\
-\Sigma(x_2x_3)c_{32} - \frac{[\Sigma(x_2x_3)]^2}{\Sigma(x_2^2)}c_{33} &= 0 \\
\hline
c_{33}\left[\Sigma(x_3^2) - \frac{[\Sigma(x_2x_3)]^2}{\Sigma(x_2^2)}\right] &= 1
\end{aligned}$$

and

$$\begin{aligned}
\frac{1}{c_{33}} &= \Sigma(x_3^2) - \frac{[\Sigma(x_2x_3)]^2}{\Sigma(x_2)^2} \\
&= \Sigma x_3^2\left[1 - \frac{[\Sigma(x_2x_3)]^2}{\Sigma(x_2)^2\Sigma(x_3)^2}\right] \\
&= n\sigma_3^2(1 - r_{23}^2)
\end{aligned}$$

Hence

$$c_{33} = \frac{1}{n\sigma_3^2(1 - r_{23}^2)}$$

Substituting this value for c_{33} in equation (91), we obtain

$$\begin{aligned}\sigma_{b_{13.2}} &= \bar{S}_{1.23}\sqrt{c_{33}} \\ &= \bar{S}_{1.23}\sqrt{\frac{1}{n\sigma_3^2(1 - r_{23}^2)}} \\ &= \sqrt{\frac{\bar{S}^2_{1.23}}{n\sigma_3^2(1 - r_{23}^2)}}\end{aligned}$$

This is seen to be identical with the value given by equation (74), when written for the corresponding coefficient. The equations to determine c_{22} for a problem of three independent variables are

$$\begin{aligned}\Sigma(x_2^2)c_{22} + \Sigma(x_2x_3)c_{23} + \Sigma(x_2x_4)c_{24} &= 1 \\ \Sigma(x_2x_3)c_{22} + \Sigma(x_3^2)c_{23} + \Sigma(x_3x_4)c_{24} &= 0 \\ \Sigma(x_2x_4)c_{22} + \Sigma(x_3x_4)c_{23} + \Sigma(x_4^2)c_{24} &= 0\end{aligned}$$

If these equations are solved simultaneously, it will be found that the value for c_{22} will be

$$c_{22} = \frac{1}{n\sigma_2^2(1 - R^2_{2.34})}$$

and substituting this in equation (91) will again show that result to be identical with equation (74). This same proof may be carried through for any number of independent variables.

APPENDIX 3

GRAPHIC CHARTS FOR INTERPRETING OR ADJUSTING CORRELATION CONSTANTS

Reliability of Small Samples.—The accompanying figures will serve to facilitate and simplify many of the computations which are discussed in the text.

Figure *A* is an extension of Table A, in Chapter 2. For samples of varying sizes, it gives the probability that the mean shown by the sample departs from the true mean of the universe by more than the stated multiple of the standard error of the mean, as computed from the sample. The figure is drawn for 2, 3, 4, 5, 6, 7, 8, 10, 12, 16, 20, 30 and ∞ observations, and may be used for any desired multiple of the standard error from 1.0 to 6.0. By interpolation, values for intermediate sizes of samples may be read. The figure is read by entering with the desired multiple of the standard error (shown at the bottom) and noting the ordinate where the line for the given number of observations intersects that abscissa. The ordinate then gives the probability of such a departure occurring solely by chance. The figure may also be entered with a desired probability and the given number of observations, to determine what multiple of the standard error must be taken to give that degree of reliability. Thus if with 10 observations a reliability of 0.95 was desired, the figure indicates 2.26 times the standard error. That is, with 10 observations, there are only 5 chances out of 100 that the true mean will not come within the range covered by observed mean ±2.26 S.E., if the sample was drawn under the conditions assumed in simple sampling.

Just as with Table A, Figure *A* may be used to judge the reliability of other coefficients than means, by subtracting 1 from the number of observations for each additional degree of freedom removed in determining the constant. For coefficients of simple correlation and regression, 1 must be subtracted; for partial or multiple correlations and regressions, subtract the number of independent variables from n; for curvilinear correlations, subtract $(m - 1)$.

Figure *A* is based upon the results given by "Student" in his article, New Tables for Testing the Significance of Observations, Metron V, No. 3, pp. 105–120. 1925.

Reliability of Observed Correlations.—Figures *B*, *C*, *D*, and *E* have been discussed in Chapter 18, pages 257 to 260. These figures provide a ready means of judging the probable minimum value for the correlation in the universe, with any observed value and any given size of sample. Under conditions of simple sampling, the chances are 19 out of 20 that the true correlation in the universe is at least as high as that given in the chart. The chart is entered with the observed correlation as abscissa; the ordinate for the intersection of that abscissa with the curve for the given size of sample gives the probable correlation. Thus if a coefficient of simple correlation, $r_{xy} = 0.65$, is obtained from a sample of 22 cases, the chances are 95 out of 100 that the true correlation is at least 0.38, according to Figure *B*. Figure *C* applies to $R_{1\cdot234}$; Figure *D* to $R_{1\cdot23456}$; and Figure *E* to $R_{1\cdot2345678}$. Values for 2, 4, and 6 independent variables may be obtained by interpolation. The figures are based upon the

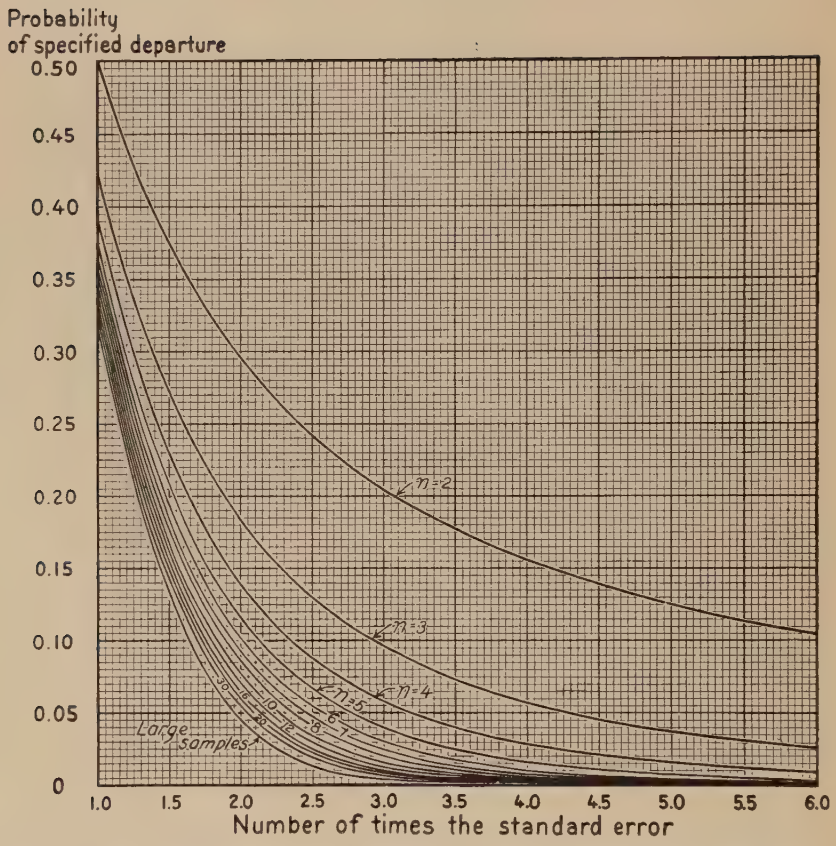

FIG. A.—The probability that an average (or other constant) computed from a sample lies further from the true value of the constant than a stated number of times the computed standard error, for samples with 2, 3, 4, 5, 6, 7, 8, 10, 12, 16, 20, 30, or ∞ observations.

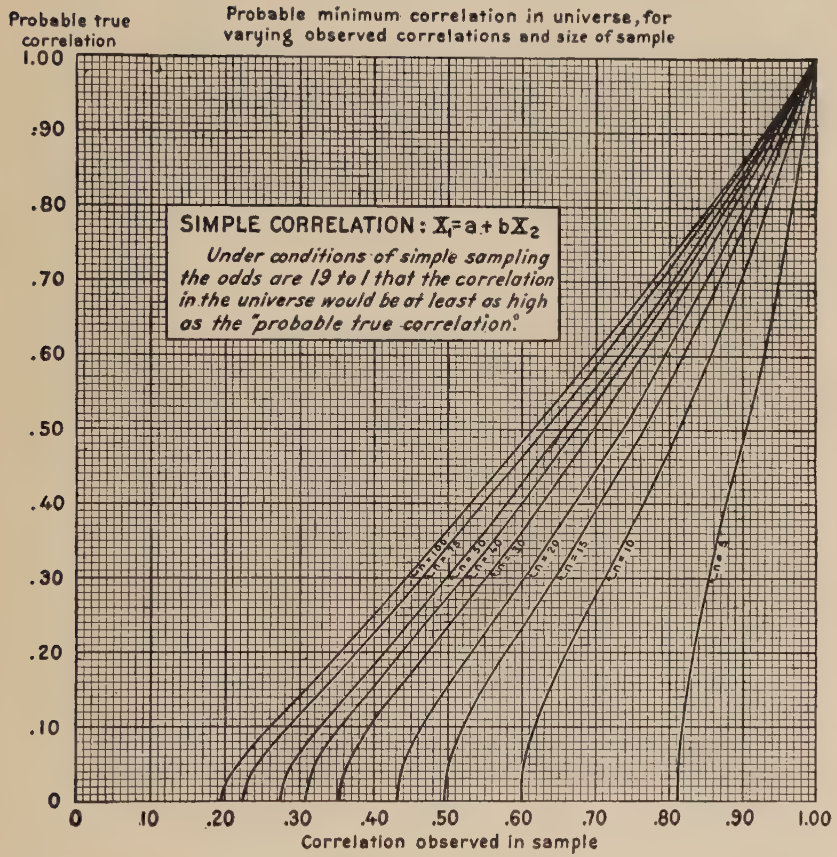

Fig. B.—The probable minimum correlation existing in the universe from which the sample was drawn, for samples of various sizes, and for varying observed correlations, for simple correlation.

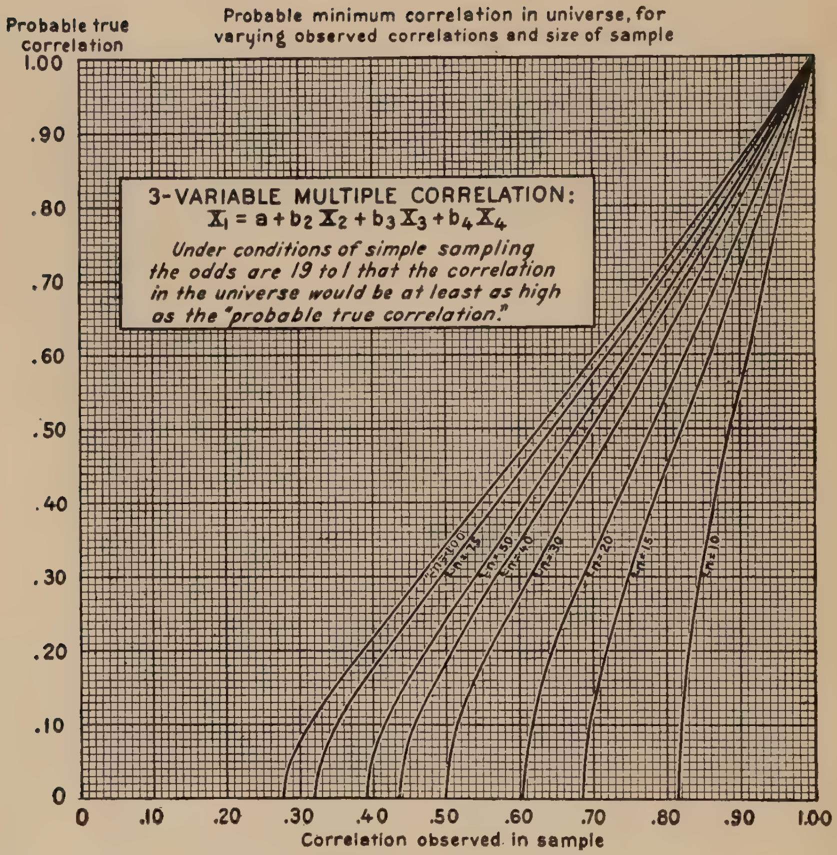

FIG. C.—The probable minimum correlation existing in the universe from which the sample was drawn, for samples of various sizes, and for varying observed correlations in the case of multiple correlation with three independent variables.

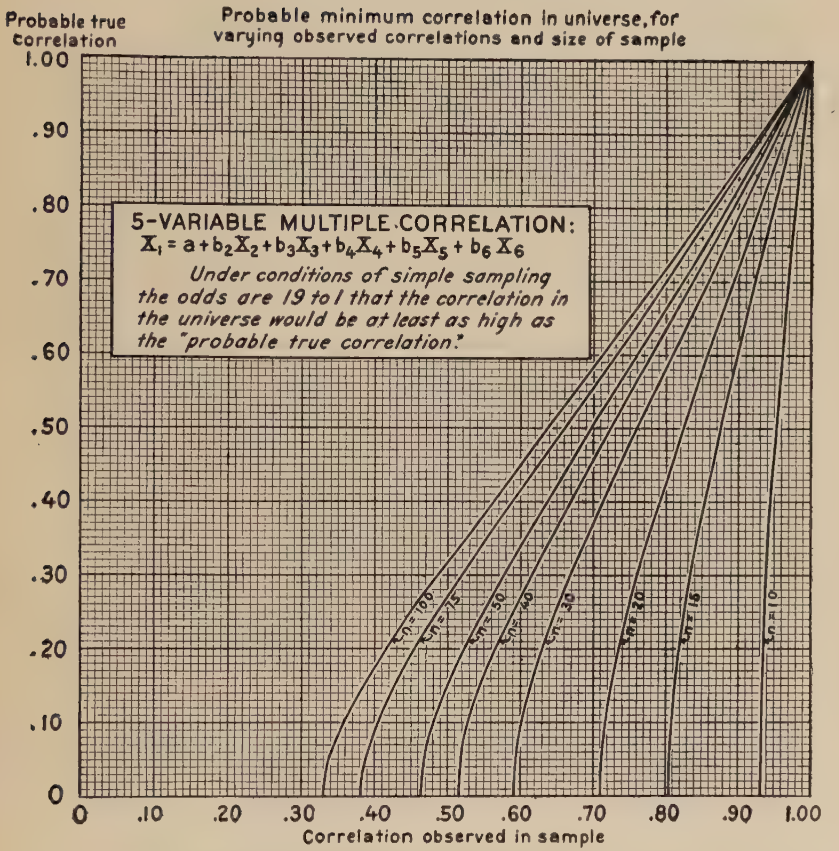

FIG. D.—The probable minimum correlation existing in the universe from which the sample was drawn, for samples of various sizes, and for varying observed correlations, in the case of multiple correlation with five independent variables.

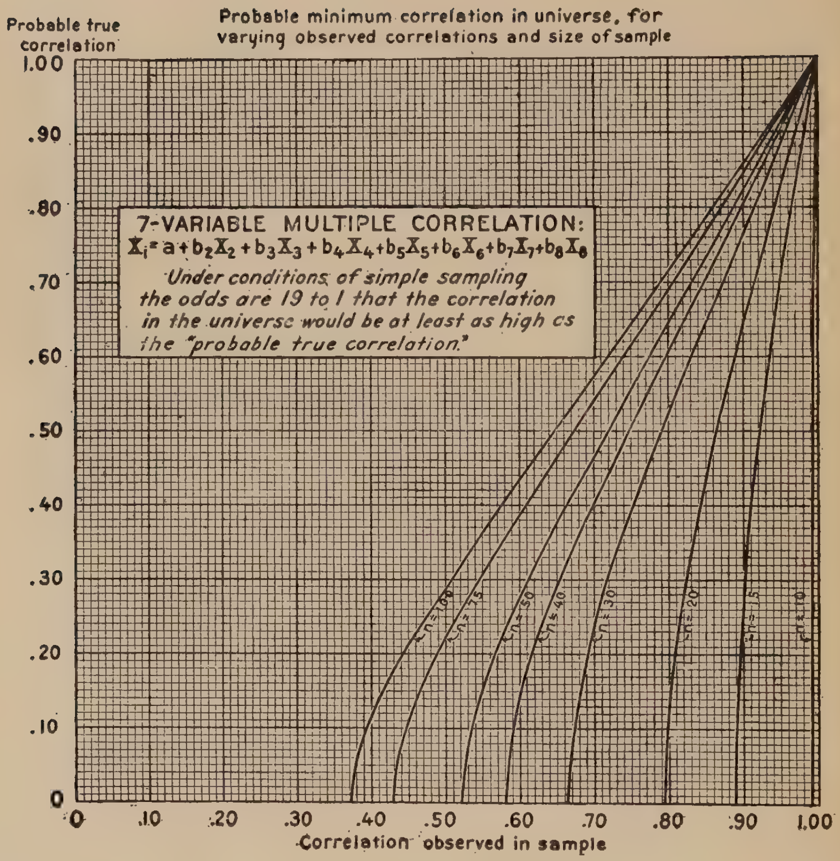

Fig. E.—The probable minimum correlation existing in the universe from which the sample was drawn, for samples of various sizes, and for varying observed correlations, in the case of multiple correlation with seven independent variables.

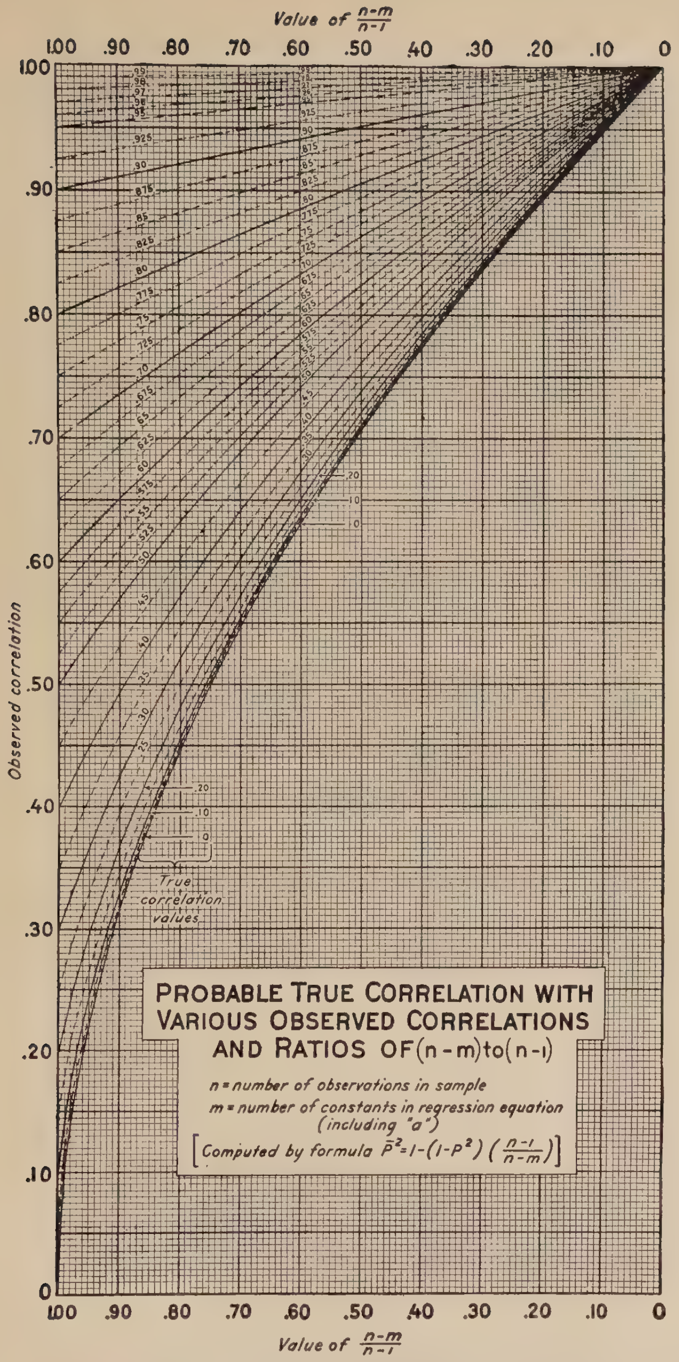

FIG. F.—This chart provides a graphic means of calculating the adjusted coefficient or index of correlation, as shown in formulas (25), (26), (47), and (66).

researches of R. A. Fisher, summarized in his recent publication "The General Sampling Distribution of the Multiple Correlation Coefficient," Proc. Royal Society, A. Vol. 121, pp. 655–673, 1928. The tables in that article assume a large sample, and therefore give only approximately correct values when applied to small samples. For that reason the values shown by Figures *B* to *E* do not agree precisely with the exact values given in the corresponding tables in Fisher's text or in Wishart's tables (cited on page 406). The differences are so slight, however, that Figures *B* to *E* are quite adequate for practical purposes.

Adjustment of Correlation for Size of Sample.—Figure *F* may be used to facilitate the calculation of adjusted coefficients and indexes of correlation, $\bar{r}$, $\bar{R}$, $\bar{\rho}$, or $\bar{\mathrm{P}}$, from the unadjusted values r, R, ρ, or P. All that is necessary is to calculate the ratio $\frac{n - m}{n - 1}$, and enter the chart with that as abscissa and the observed correlation as ordinate. Then note the value of $\bar{r}$ given by the curve which lies nearest the intersection of the two coordinates, or interpolate between the two nearest curves. Thus with $\mathrm{P} = 0.70$, $n = 30$, and $m = 8$, $\frac{n - m}{n - 1} = 0.759$, and $\bar{\mathrm{P}} = 0.57$. Likewise for $R_{1.234} = 0.88$, and $n = 20$; $\frac{n - m}{n - 1} = \frac{16}{19} = 0.842$, and $\bar{R}_{1.234} = 0.85$.

APPENDIX 4

LIST OF IMPORTANT EQUATIONS

For convenience in referring to the most important of the equations which are introduced from time to time in the text, all numbered equations are repeated here in numerical order.

$$M_x = \frac{\Sigma X}{n} \tag{1}$$

$$X - M_x = x \tag{2}$$

$$\delta = \frac{\Sigma x \text{ (without regard to sign)}}{n} \tag{3}$$

$$\sigma_x = \sqrt{\frac{\Sigma x^2}{n}} \tag{4}$$

$$\sigma_x = \sqrt{\frac{\Sigma(X^2)}{n} - n(M_x)^2} \tag{5}$$

$$\sigma_u = \sqrt{\frac{\Sigma(d^2F)}{n} - \left[\frac{\Sigma(dF)}{n}\right]^2 - \frac{c}{12}} \tag{6}$$

$$\sigma_M = \frac{\sigma_x}{\sqrt{n-1}} \tag{7}$$

$$Y = a + bX \tag{8}$$

$$b_{yx} = \frac{\Sigma(XY) - nM_xM_y}{\Sigma(X^2) - n(M_x)^2} \tag{9}$$

$$a = M_y - bM_x \tag{10}$$

$$\Sigma(XY) - nM_xM_y = \Sigma(xy) \tag{11}$$

$$Y = a + bX + cX^2 \tag{12}$$

[Using X for X, x for $X - M_x$, U for X^2, u for $U - M_u$, equation (12) becomes $Y = a + bX + cU$.]

$$\left.\begin{aligned} \Sigma(x^2)b + \Sigma(xu)c &= \Sigma(xy) \\ \Sigma(xu)b + \Sigma(u^2)c &= \Sigma(uy) \end{aligned}\right\} \tag{13}$$

$$a = M_y - b(M_x) - c(M_u) \tag{14}$$

$$\left.\begin{aligned}
M_x &= \frac{\Sigma(X)}{n};\ M_u = \frac{\Sigma(U)}{n};\ M_y = \frac{\Sigma(Y)}{n} \\
\Sigma(x^2) &= \Sigma(X^2) - n(M_x)^2 \\
\Sigma(xu) &= \Sigma(XU) - n(M_xM_u) \\
\Sigma(u^2) &= \Sigma(U^2) - n(M_u)^2 \\
\Sigma(xy) &= \Sigma(XY) - n(M_xM_y) \\
\Sigma(uy) &= \Sigma(UY) - n(M_uM_y)
\end{aligned}\right\} \tag{15}$$

$$Y = a + bX + cX^2 + dX^3 \tag{16}$$

[Using U for X^2, V for X^3, eq. (16) becomes $Y = a + bX + cU + dV$.]

$$\left.\begin{aligned}
\Sigma(x^2)b + \Sigma(xu)c + \Sigma(xv)d &= \Sigma(xy) \\
\Sigma(xu)b + \Sigma(u^2)c + \Sigma(uv)d &= \Sigma(uy) \\
\Sigma(xv)b + \Sigma(uv)c + \Sigma(v^2)d &= \Sigma(vy)
\end{aligned}\right\} \tag{17}$$

$$a = M_y - b(M_x) - c(M_u) - d(M_v) \tag{18}$$

$$\left.\begin{aligned}
M_v &= \frac{\Sigma(V)}{n} \\
\Sigma(uv) &= \Sigma(UV) - n(M_uM_v) \\
\Sigma(xv) &= \Sigma(XV) - n(M_xM_v) \\
\Sigma(v^2) &= \Sigma(V^2) - n(M_v)^2
\end{aligned}\right\} \tag{19}$$

$$\left.\begin{aligned}
S_{yx}{}^2 &= \frac{\Sigma z^2}{n-1} \\
S^2{}_{y \cdot f(x)} &= \frac{\Sigma(z'')^2}{n-1}
\end{aligned}\right\} \tag{20}$$

$$r_{yx} = \frac{\sigma_{y'}}{\sigma_y} \tag{21}$$

$$\rho_{yx} = \frac{\sigma_{y''}}{\sigma_y} \tag{22}$$

$$\bar{S}^2{}_{yx} = \frac{n\sigma_z{}^2}{n-2} \tag{23}$$

$$\bar{S}^2{}_{y \cdot f(x)} = \frac{n\sigma^2{}_{z''}}{n-m} \tag{24}$$

$$\bar{r}^2{}_{yx} = 1 - (1 - r^2)\left(\frac{n-1}{n-2}\right) \tag{25}$$

$$\bar{\rho}^2{}_{yx} = 1 - (1 - \rho^2)\left(\frac{n-1}{n-m}\right) \tag{26}$$

$$r_{xy} = \frac{\Sigma(XY - nM_xM_y)}{\sqrt{[\Sigma(X^2) - n(M_x)^2]\,[\Sigma(Y^2) - n(M_y)^2]}} \tag{27}$$

$$\overline{S}_{yx} = \sqrt{\frac{\Sigma(Y^2) - n(M_y)^2}{n - 1}(1 - \overline{r}^2{}_{xy})} \tag{28}$$

$$\overline{\rho}^2{}_{yx} = 1 - \left(\frac{\sigma^2{}_{z''}}{\sigma_y{}^2}\right)\left(\frac{n - 1}{n - m}\right) \tag{29}$$

$$X_1 = a + b_2X_2 + b_3X_3 \tag{30}$$

$$\left.\begin{aligned} \Sigma(x_2{}^2)b_2 \;+ \Sigma(x_2x_3)b_3 &= \Sigma(x_1x_2) \\ \Sigma(x_2x_3)b_3 + \Sigma(x_3{}^2)b_3 \;&= \Sigma(x_1x_3) \end{aligned}\right\} \tag{31}$$

$$a = M_1 - b_2M_2 - b_3M_3 \tag{32}$$

$$X'_1 = a + b_2X_2 + b_3X_3 \tag{33}$$

$$z = X_1 - X'_1 \tag{34}$$

$$X_1 = a_{1.23} + b_{12.3}X_2 + b_{13.2}X_3 \tag{35}$$

$$X_1 = a_{1.234} + b_{12.34}X_2 + b_{13.24}X_3 + b_{14.23}X_4 \tag{36}$$

$$X_1 = a_{1.2345} + b_{12.345}X_2 + b_{13.245}X_3 + b_{14.235}X_4 + b_{15.234}X_5 \tag{37}$$

$$\left.\begin{aligned} \Sigma(x_2{}^2)b_{12.34} \;+ \Sigma(x_2x_3)b_{13.24} + \Sigma(x_2x_4)b_{14.23} &= \Sigma(x_1x_2) \\ \Sigma(x_2x_3)b_{12.34} + \Sigma(x_3{}^2)b_{13.24} \;+ \Sigma(x_3x_4)b_{14.23} &= \Sigma(x_1x_3) \\ \Sigma(x_2x_4)b_{12.34} + \Sigma(x_3x_4)b_{13.24} + \Sigma(x_4{}^2)b_{14.23} \;&= \Sigma(x_1x_4) \end{aligned}\right\} \tag{38}$$

$$a_{1.234} = M_1 - b_{12.34}M_2 - b_{13.24}M_3 - b_{14.23}M_4 \tag{39}$$

$$\left.\begin{aligned} \Sigma(x_2{}^2)b_{12.345} \;+ \Sigma(x_2x_3)b_{13.245} &+ \Sigma(x_2x_4)b_{14.235} \\ &+ \Sigma(x_2x_5)b_{15.234} = \Sigma(x_1x_2) \\ \Sigma(x_2x_3)b_{12.345} + \Sigma(x_3{}^2)b_{13.245} \;&+ \Sigma(x_3x_4)b_{14.235} \\ &+ \Sigma(x_3x_5)b_{15.234} = \Sigma(x_1x_3) \\ \text{Etc.} \qquad & \end{aligned}\right\} \tag{40}$$

$$a_{1.2345} = M_1 - b_{12.345}M_2 - b_{13.245}M_3 - b_{14.235}M_4 - b_{15.234}M_5 \tag{41}$$

$$\overline{S}^2{}_{1.234} = \frac{n\sigma^2{}_{z_{1.234}}}{n - m} \tag{42}$$

$$\overline{S}^2{}_{1.234\ldots n} = \frac{\left\{\begin{matrix} \Sigma(x_1{}^2) - [b_{12.34\ldots n}(\Sigma x_1x_2) + b_{13.24\ldots n}(\Sigma x_1x_3) \\ + \ldots + b_{1n.23\ldots(n-1)}(\Sigma x_1x_n)] \end{matrix}\right\}}{n - m} \tag{43}$$

$$X_{1(234)} = a_{1.234} + b_{12.34}X_2 + b_{13.24}X_3 + b_{14.23}X_4 \tag{44}$$

$$R_{1.234} = \frac{\sigma_{1(234)}}{\sigma_1} \tag{45}$$

$$R^2{}_{1.234\ldots n} = \frac{\left\{\begin{aligned} b_{12.34\ldots n}(\Sigma x_1x_2) + b_{13.24\ldots n}(\Sigma x_1x_3) + \ldots \\ + b_{1n\ldots 23\ldots(n-1)}(\Sigma x_1x_n) \end{aligned}\right\}}{\Sigma(x_1{}^2)} \tag{46}$$

$$\overline{R}^2{}_{1.234\ldots n} = 1 - (1 - R^2{}_{1.234\ldots n})\left(\frac{n-1}{n-m}\right) \tag{47}$$

$$\overline{R}^2{}_{1.234\ldots n} = 1 - \left(\frac{\overline{S}^2{}_{1.234\ldots n}}{\sigma_1{}^2}\right)\left(\frac{n-1}{n}\right) \tag{48}$$

$$\overline{S}^2{}_{1.234\ldots n} = \sigma_1{}^2(1 - \overline{R}^2{}_{1.234\ldots n})\left(\frac{n}{n-1}\right) \tag{49}$$

$$\bar{r}^2{}_{14.23} = 1 - \frac{(1 - \overline{R}^2{}_{12.34})}{(1 - \overline{R}^2{}_{1.23})} \tag{50}$$

$${}_{12}\bar{r}^2{}_{34} = \frac{b^2{}_{12.34}\sigma_2{}^2}{b^2{}_{12.34}\sigma_2{}^2 + \sigma_1{}^2(1 - \overline{R}^2{}_{1.234})} \tag{51}$$

$$\beta_{12.34} = b_{12.34}\frac{\sigma_2}{\sigma_1} \tag{52}$$

$$\left.\begin{array}{l}\text{Multiple correlation of} \\ (X_1 - b_{12.34}X_2)\text{ with} \\ X_3\text{ and }X_4\end{array}\right\} = 1 - \frac{\sigma_1{}^2(1 - R^2{}_{12.34})}{\sigma_1{}^2 - 2b_{12.34}\left(\frac{\Sigma x_1x_2}{n}\right) + b^2{}_{12.34}\sigma_2{}^2} \tag{53}$$

$$X_1 = a + f(X_2) + f(X_3) + f(X_4) + \ldots \tag{54}$$

$$\left.\begin{aligned} X_1 = a + b_2X_2 + b_{2'}(X_2{}^2) + b_3X_3 + b_{3'}(X_3{}^2) \\ + b_4X_4 + b_{4'}(X_4{}^2) \end{aligned}\right\} \tag{55}$$

$$\left.\begin{aligned} X_1 = a + b_2(X_2) + b_{2'}(X_2{}^2) + b_{2''}(X_2{}^3) + b_3(X_3) + b_{3'}(X_3{}^2) \\ + b_{3''}(X_3{}^3) + b_4(X_4) + b_{4'}(X_4{}^2) + b_{4''}(X_4{}^3) \end{aligned}\right\} \tag{56}$$

$$X''_1 = a'_{1.234} + f'(X_2) + f'(X_3) + f'(X_4) \tag{57}$$

$$a'_{1.234} = M_1 - \frac{\Sigma[f'(X_2) + f'(X_3) + f'(X_4)]}{n} \tag{58}$$

$$z'' = X_1 - X''_1 \tag{59}$$

$$X'_1 = F(X_2) = f(X_2) - M_{f(2)} + M_1 \tag{60}$$

$$x'_1 = F(x_2) = f(X_2) - M_{f(2)} \tag{61}$$

$$X'_1 = F(x_2) + F(X_3) + F(x_4) \ldots + F(x_n) \tag{62}$$

$$S_{1.f(2,3,4)} = \sigma_{z_{1.f(2,3,4)}} \tag{63}$$

$$\overline{S}^2{}_{1.f(2,\,3,\,4,\,\text{etc.})} = \frac{\sigma^2{}_{z_{1.f(2,\,3,\,4,\,\text{etc.})}}}{1 - m/n} \tag{64}$$

$$\overline{S}^2{}_{1.f(2,\,3,\,4,\,\text{etc.})} = \frac{n\sigma_z{}^2}{n-m} = \frac{\Sigma(z^2)}{n-m} \tag{65}$$

$$\left.\begin{aligned}\overline{\mathrm{P}}^2{}_{1.234} &= 1 - \frac{\overline{S}^2{}_{1.f(2,\,3,\,4)}}{\sigma_1{}^2}\left(\frac{n-1}{n}\right)\\ \overline{\mathrm{P}}^2{}_{1.234} &= 1 - \frac{\sigma^2{}_{z_{1.f(2,\,3,\,4)}}}{\sigma_1{}^2}\left(\frac{n-1}{n-m}\right) = 1 - \frac{\Sigma(z^2{}_{1.f(2,\,3,\,4)})}{\Sigma(x_1{}^2)}\left(\frac{n-1}{n-m}\right)\end{aligned}\right\} \tag{66}$$

$$\overline{\mathrm{P}}^2{}_{1.234} = 1 - (1 - R^2{}_{1.2'3'4'})\left(\frac{n-1}{n-m}\right) \tag{67}$$

$$\eta_{yx} = \sqrt{\frac{\Sigma[n_0(M_0)^2] - n(M_y)^2}{n\sigma_y{}^2}} \tag{68}$$

$$\sigma_{b_{yx}} = \frac{\overline{S}_{y.x}}{\sigma_x\sqrt{n}} \tag{69}$$

$$\sigma_{M_{y'}} = \frac{\overline{S}_{y.x}}{\sqrt{n}} \tag{70}$$

$$\sigma_{r_{yx}} = \frac{1-r^2}{\sqrt{n-2}}, \text{ for large values of } n \tag{71}$$

$$\sigma_{\rho_{xy}} = \frac{1-\rho^2}{\sqrt{n-m}}, \text{ for large values of } n \tag{72}$$

$$\sigma_{R_{1.234}} = \frac{1-R^2{}_{1.234}}{\sqrt{n-m}}, \text{ for large values of } n \tag{73}$$

$$\sigma_{b_{12.34}} = \sqrt{\frac{\overline{S}^2{}_{1.234}}{\sigma_2{}^2(1-R^2{}_{2.34})n}} \tag{74}$$

$$X_1 = f(X_2, X_3) \tag{75}$$

$$X_1 = f(X_2, X_3) + f(X_4) \tag{76}$$

$$X_1 = f(X_2, X_3) + f(X_4, X_5) + f(X_6) \tag{77}$$

$$X_1 = f(X_2, X_3, X_4, \ldots X_n) \tag{78}$$

$$z'''' = a_{z.234} + b_{z2.34}X_2 + b_{z3.24}X_3 + b_{z4.23}X_4 \tag{79}$$

$$\left.\begin{aligned}X_1 = a_{1.2'3'4'} + b_{12'.3'4'}\,[f'''(X_2)] + b_{13'.2'4'}\,[f'''(X_3)]\\ + b_{14'.2'3'}\,[f'''(X_4)]\end{aligned}\right\} \tag{80}$$

$$X''''_1 = \theta(X'''_1) \tag{81}$$

$$X''''_1 = \theta\,[a + f'''(X_2) + f'''(X_3) + f'''(X_4)] \tag{82}$$

$$X_1 = a + eX_3 + g(X_2X_3) \tag{83}$$

$$X_1 = a + eX_3 + g(X_2X_3) + h(X_4) \tag{84}$$

$$X_1 = a + f(X_2) + f(X_2X_3) + f(X_4) \tag{85}$$

$$X_1 = f(X_2) + f(X_3) + f(X_2 + X_3) \tag{86}$$

$$X_1 = f(X_2) + f(X_3) + f\left(\frac{X_2}{a} + \frac{X_3}{b}\right) + f\left(\frac{X_2}{a} - \frac{X_3}{c}\right) \tag{87}$$

$$\left.\begin{aligned} X_1 = f(X_2) + f(X_3) + f(X_4) & \\ + f\left(\frac{X_2}{\sigma_2} + \frac{X_3}{\sigma_3} + \frac{X_4}{\sigma_4}\right) + f\left(\frac{X_2}{\sigma_2} + \frac{X_3}{\sigma_3} - \frac{X_4}{\sigma_4}\right) & \\ + f\left(\frac{X_2}{\sigma_2} - \frac{X_3}{\sigma_3} + \frac{X_4}{\sigma_4}\right) + f\left(\frac{X_2}{\sigma_2} - \frac{X_3}{\sigma_3} - \frac{X_4}{\sigma_4}\right) & \end{aligned}\right\} \tag{88}$$

$$R_{1.2,\,2^2,\,2^3,\,3,\,3^2,\,3^3,\,\ldots\,n,\,n^2,\,n^3} = \mathrm{P}_{1.23\ldots n} \tag{89}$$

$$\overline{S}^2_{1.f(23\ldots n)} = \sigma_1{}^2(1 - \overline{\mathrm{P}}^2_{1.23\ldots n})\left(\frac{n}{n-1}\right) \tag{90}$$

$$\left.\begin{aligned} \sigma_{b_{12.34}} &= \overline{S}_{1.234}\sqrt{c_{22}} \\ \sigma_{b_{13.24}} &= \overline{S}_{1.234}\sqrt{c_{33}} \\ \sigma_{b_{14.23}} &= \overline{S}_{1.234}\sqrt{c_{44}} \end{aligned}\right\} \tag{91}$$

APPENDIX 5

GLOSSARY

The Greek letters used as symbols in this text, and the most important other symbols, are as follows:

δ (small *delta*) = coefficient of average variation.
σ (small *sigma*) = coefficient of standard variation.
Σ (capital *sigma*) = sum of the items specified.
n (Latin) = number of observations in a sample.
b (Latin) = coefficient of regression.
$f(\,)$ (Latin) = function of the variable in the parenthesis.
r (Latin) = coefficient of correlation.
ρ (small *rho*) = index of (curvilinear) correlation.
S (Latin) = standard error of estimate.
m (Latin) = number of constants in the regression equation.
z (Latin) = residual, or difference between observed and estimated values of a dependent variable.
R (Latin) = coefficient of multiple correlation.
β (small *beta*) = " beta " coefficient of regression, in terms of unit standard deviations.
P (capital *rho*) = index of multiple (curvilinear) correlation.
η (small *eta*) = correlation ratio.
θ (small *theta*) = function of (used here for the Bruce adjustment function).
Δ (capital *delta*) = arbitrary symbol.
π (small *pi*) = arbitrary symbol.
Φ (capital *phi*) = function of.

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Excellent bibliographies covering the basic development of the theory of statistics are given in G. U. Yule's "Introduction to the theory of statistics," and in the recent special study, "The history of statistical method," by Helen M. Walker. In addition, a brief list of references covering especially articles on the theory of sampling is given in R. A. Fisher's "Statistical methods for research workers." No attempt will be made here to repeat these bibliographies; instead, the student of statistical theory is referred to the sources mentioned.

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